The Arithmetic Teacher

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THE ARITHMETIC TEACHER

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Arithmetic for the Academically Talented

The Arithmetic Program in the Member Independent Schools of the Educational Records Bureau

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In his recent report on the American High School Dr. Conant[1] gives careful consideration to the comprehensive-type school, for he is of the opinion that this type of school satisfies the needs of the greater part of the nation's youth. However, he also gives proper recognition to other types of schools in our society: "I cannot emphasize too strongly the differences I have found between industrial cities, suburban areas, and the districts I have visited in certain large cities . . . improvements must come school by school and be made with due regard for the nature of the community."[1] In his discussion of the schools found in suburban districts—and he includes certain metropolitan schools in this category—he mentions two chief characteristics. First: the students in these schools are markedly superior in academic ability as measured by scholastic aptitude tests; and secondly: the majority of these students are preparing for college.

With the above facts in mind we can state that in these communities are also found elementary schools containing students of the same type that feed the high schools of the area. These students, too, are superior in ability, and in most cases college-bound. That such students learn more rapidly than the average has often been shown by research; studies also show that in some foreign countries even the average student seems to cover more ground in less time than that allotted in our country. Even before the launching of Sputnik the author of this study began to wonder if any attempts were being made in the United States to give the above-average child a more challenging arithmetic program. It would seem that schools like those described above, where there are large concentrations of academically talented youth, would provide the most information concerning an arithmetic program appropriate to their needs. Although these schools are basically of two different types—public and independent—in many instances they find a common meeting ground in the testing services of Educational Records Bureau. This organization, incorporated and chartered by the Board of Regents of the University of the State of New York, is a non-profit-making service and research agency for schools and colleges.[2] It was organized in 1927 to assist member schools in securing reliable and comparable measurements of the aptitudes and academic achievements of

[1] J. B. Conant, *The American High School* (New York: McGraw-Hill Book Company, 1959), p. 95.

[2] Educational Records Bureau, 21 Audubon Ave., New York 32, N.Y.

their pupils. All E.R.B. test-results are converted into percentiles; thus the Bureau's norms enable a school to analyze: (a) a pupil's performance on different tests in terms of a single scoring system based on a relatively *constant college-preparatory student population*, and (b) his *achievement in relation to his ability*, and also to eliminate the danger of invalid conclusions based only on local percentiles.

Since its organization in 1927 the membership of the Bureau has increased from the first year's total of 53 schools to include 794 institutions in June, 1959. Public schools are included in this total and they find the services of the Bureau increasingly helpful, especially those schools that provide special classes for the gifted and academically talented and those designated by Conant as "surburban." Membership in the Bureau affords access to norms which allow the analysis of the achievement of college-preparatory pupils in public schools in terms of test-results in a group of schools enrolling pupils of like caliber. The over-all logical advantage for all schools using this testing service is that students can know from the very beginning of their school career how they measure up with other students, who will in most part form their associates in college work.

Since reliable statistics gathered by the E.R.B. over a period of twenty-five years were available, it seemed profitable to make use of them for a study of the arithmetic program in the member schools of the organization. It was hoped, too, that the unifying factors of a common testing program, various administrative devices and over-all goals would lead to a certain agreement as to educational philosophy, content, and methods.

With the approval of Dr. A. Traxler, executive director of the Bureau, a seventeen-page questionnaire covering all phases of the arithmetic program was sent to 500 member schools in the fall of 1957. One hundred and fifteen complete forms were returned. Considering the length of the questionnaire and the fact that some schools had few elementary classes, this seemed a satisfactory return. In broad outline the study was designed to answer the following questions:

1. What administrative devices seem most appropriate for the instruction of the academically-talented in arithmetic?
2. What curriculum, teaching materials, and methods are being used in the arithmetic programs in E.R.B. schools, and what are the opinions of the instructors as to their effectiveness?

The statistics and opinions received from the questionnaire were sorted and studied over a period of 6 months. A summary of the results follows.[3]

Grouping and Class Size

One of the first administrative policies to be decided upon is the type of class organization—both as to grouping and as to class size. Within the E.R.B. group of schools, certain factors have a determining effect on the type of grouping to be chosen. As Conant remarked, and as the table (p. 55) shows, children in these schools are in general above-average in intelligence. Research has revealed that children from upper socio-economic groups tend to be above-average, and this characteristic is somewhat accentuated in many of these schools by entrance requirements.[4] Within the past few years college entrance requirements have stiffened, and this in turn has had its effects on the elementary and secondary college-preparatory schools. The following table gives an indication of the intelligence level in E.R.B. schools.

[3] For detailed information concerning this study see "An Arithmetic Program for the Above-Average Child" (Unpublished M.A. thesis, Department of Education, St. Louis University, 1959), pp. 200.

[4] A. Anastasi and J. P. Foley, *Differential Psychology* (New York: The Macmillan Company, 1949), pp. 787–830.

MEDIAN INTELLIGENCE QUOTIENTS OF CLASSES TESTED IN E.R.B. INDEPENDENT SCHOOLS.
SEPTEMBER 15 to NOVEMBER 1, 1957—KUHLMAN ANDERSON 6th ED.

Grade	1	2	3	4	5	6	7	8
Median	122.7	122.6	122.8	122.8	127.7	121.5	123.8	120.3

When we learn that the majority of schools (80 to 90 per cent) designated heterogeneous grouping as the type most often employed in the organization of their classes, we must remember that the entire student body is already selective upon entrance and that the various class groups are in actual fact homogeneous with regard to ability. In the first five elementary grades, 50% of the schools received only enough children to form a single class, while the other 50% of these and the remaining grades, 6, 7, and 8, divided those enrolled into smaller classes of 15 to 25 children. Many instructors stated that the small class was essential if individual attention and rates of progress were to be given proper recognition. The exact statistics are given below.

Time Allotments for Instruction and Study Period

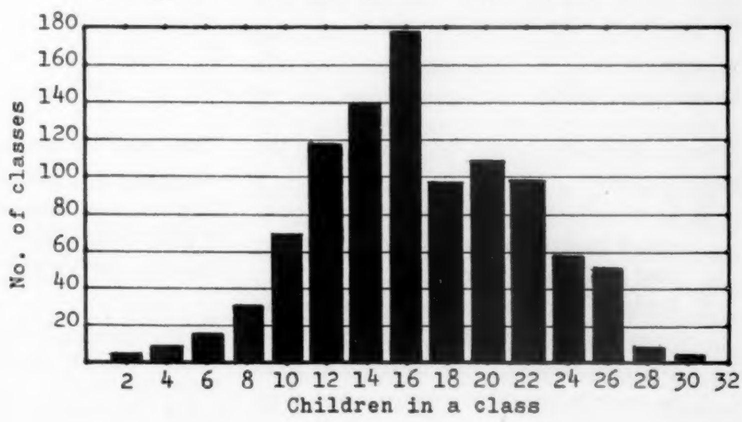
Besides the method of grouping, another important administrative factor is the amount of time provided for the class and study period. Although many instructors expressed a desire for a longer arithmetic period, in most cases they were granted only the conventional time-allotments, ranging from 20 to 30 minutes a day for the 1st and 2nd grades to 40 minutes for the remaining grades 3–8. In almost every case (95% of the classes) these were scheduled in the morning. Because of the greater number of divisions in the 7th and 8th grades, a few had to be scheduled in the afternoon.

Information received in the questionnaire makes it quite evident that over and above the daily class period, homework plays a vital role in the E.R.B. schools. Fourteen to fifteen per cent of the schools give 10 to 20 minutes a day in arithmetic in the first two grades; beginning in the 3rd grade 20 minutes or more are required by the large majority (60 to 90 per cent). This integral part of the learning process was referred to in several parts of the questionnaire-responses, where methods for assigning, forms required in organized notebooks, means used for daily correction, and sanctions for incomplete work are outlined in detail. Many

CLASS SIZES

Grade	Median Nos. of Children
1	18
2	17
3	16
4	15
5	16
6	15
7	16
8	15

Graph showing distribution of class sizes throughout all eight grade - 954 classes.



felt that the consistent requirement of homework is not only a strengthening factor in the arithmetic program itself, but a powerful means for training in responsibility—especially in the upper grades, where long-range assignments are given over a period of two or three weeks.

Teaching Experience of the Instructors

One other administrative factor that seems to have some bearing on the successful instruction of the academically talented is the number of years of teaching experience of the instructors. In a few instances beginning teachers were employed, but in the large majority of cases only experienced instructors were being used. In the lower grades the instruction in arithmetic was one of many other subjects assigned to a particular teacher who was often with the same class all day, but in the upper grades, where the instructors were often men, it was found to be most efficacious to have one instructor take a class through two or three years of arithmetic study. In this way the strengths and weaknesses of each child were more easily followed, and there was greater opportunity for seeing that those capable of more advanced work were given the chance to progress at individual rates. The exact figures are given below.

TEACHING EXPERIENCE OF INSTRUCTORS IN E.R.B. SCHOOLS

Grade	1	2	3	4	5	6	7	8
Median No. of Years General Experience	15	15	14	10	14	11	11	12
Median No. of Years Teaching This Grade Arithmetic	8	7	5	4	5	5	6	7

Several inferences might be drawn from the above figures:

1) It would seem that the more experienced instructors are being used in the first and second grades.
2) Perhaps because of the particular qualities needed for teaching these younger children, the instructors tend to remain in these grades once they have been found to be successful.
3) Since so many remark (as will be seen later in the study) that weakness in arithmetic is due to insufficient mastery of basic skills, and since the majority of these remarks come from instructors in grades 5 through 8, it might be well to have more experienced teachers for the arithmetic classes in grades 3 and 4.

Although the administrative factors discussed above often have a determining effect on the possibilities for success in the instruction of arithmetic, more influential factors will be found in the choice of curriculum, instructional materials, and methods to be used.

Curriculum

The fact that children with above-average ability are able to learn more quickly than the average child has influenced many of the E.R.B. schools in the determination of their curriculum. Since the repetition of the same material in a variety of forms is a waste of time for them once the basic principles have been mastered and a sufficient number of applications have been made, classes are allowed to go ahead with some of the material of the next year. In this way there is a slight acceleration without skipping the work of any year, and enough time is gained to save in some cases a whole year of work. This type of procedure has been recommended by J. J. Gallagher[5] of the American Educational Research Association of the N.E.A. When this study was made in 1957, one-third of the E.R.B. schools were covering

[5] J. J. Gallagher, "The Gifted Child in the Elementary School," *What Research Says to the Teacher*, Series No. 17, American Ed. Research Assoc. of N.E.A. (Washington D.C.—1201 16th St. N.W.) 1959.

PERCENTAGE OF E.R.B. SCHOOLS DEVOTING THE DESIGNATED PORTIONS OF THE SCHOOL YEAR TO ALGEBRA IN THE 8TH GRADE

Whole year		One half year	Less than one half year
31%		32%	36%
23% For entire class	8% For superior group only		

Credit: 1 high-school credit given for algebra in the 8th grade 13%

(It should be noted that other schools wanted to give 1 high-school credit but were prevented from doing so by accrediting agencies.)

the eight years of arithmetic in seven, and giving a year of algebra in the eighth grade. The exact statistics are given at the top of this page.

These figures have undoubtedly increased since 1957, owing to the new emphases and approaches to the study of mathematics that are being formulated by various organizations throughout the country.[6] The Committee on Mathematics Tests of the Bureau is at present in the process of surveying the curriculum in mathematics in Grades 7 through 12 of Bureau Member schools and is inquiring into the contemplated changes in the mathematics curriculum to take account of recommendations of the College Entrance Board's Commission on Mathematics and other groups. The Committee expects to make a report of its findings during the school year 1959–1960. In view of the significance of this portion of the study it might be well to quote a few of the remarks submitted by the instructors concerning the reasons and advantages of arranging the curriculum to begin algebra in the 8th grade.

FROM THOSE GIVING THE WHOLE YEAR:

"*Enough arithmetic* has been covered in seven years *to meet the needs* of the pupils."

"Any pupil with a *good mind* should finish arithmetic in seven years."

"Children love it and are ready for it . . . *ready for* something more *abstract*."

"Having had algebra in the eighth grade, students can move more quickly in the ninth grade and *cover parts of algebra often omitted* because of lack of time."

"Two years in first year algebra give a *good grounding*."

FROM THOSE GIVING HALF A YEAR:

"Basic arithmetic *skills can be better taught* with a fresh approach through algebra."

"*Algebra can be introduced more easily* if shown in similar arithmetic problems. The relationship between the two can be brought out."

"*Essential arithmetic is exhausted* by the second semester of the eighth grade."

One other important advantage in providing a year of algebra in the 8th grade, which was perhaps not fully realized at the time this study was made, is the possibility of offering one more year of mathematics in high school. This would enable a student to try for the Advanced Placement Program.

In order to get an exact picture of the topics covered in each grade in the process of accelerating, the instructors were asked to check a detailed list covering the broad content fields in the tests as designated by the Bureau for each grade level. This information showed that a small portion of advanced content was taught in each grade. Multiplication and division were introduced in the second grade, all of the tables were

[6] *School Mathematics Study Group Newsletter No. 1* (New Haven: Yale University, 1959).

covered by the third grade, fractions were begun in the fourth, etc. Spreading the work of one year over seven made the acceleration so gradual that even the slowest students did not feel pushed. However, this acceleration did bring with it several problems, the chief one being that of textbooks.

Textbooks

That arithmetic textbooks pose a real problem for E.R.B. schools was evidenced by many remarks to the questionnaire. In order to cover some of the matter of the following year, it was necessary to use in one year not only the text designated for that grade but also part of the next book. The instructors also complained that rarely was a complete series satisfactory, so that very few found it possible to use the same series throughout the eight grades. It is true that many series do not begin until the third grade, but even taking this into consideration, the average number of changes in a series was four. The remark was repeatedly make that the texts were too easy, and that not enough drill material was given. Although many series are beginning to include more drill matter, B. F. Skinner, the educational psychologist, states: "Perhaps the most serious criticism of the current classroom is the relative infrequency of reinforcement . . . even the most modern workbook in beginning arithmetic is far from exemplifying an efficient program for shaping up mathematical behaviour."[7] In the listings of series used, the variety was so great that in only a few cases was the same author mentioned more than ten times. Two publications, however, should be cited here, not only for frequency of mention, but for the emphatic approval given by so many to their content: Catherine Stern's Structural Arithmetic materials[8] for the early grades and John A. McGuinn's Problem Books for grades 5 through 8.[9]

A few schools have resorted to compiling their own materials for the eight grades. It is hoped that in the near future, when more and more schools seek materials for the gifted child, a series will be developed for grades 1 through 6 to dovetail with the programs being prepared for the 7th and 8th grades.

Methods

The last topic to be discussed is that of methods employed in E.R.B. schools for the teaching of arithmetic. The comments brought out an important factor that has much influence on the type of methods to be chosen, namely that above-average children do not as a rule have difficulty with the understandings in arithmetic, but decidedly dislike the necessary drill and routine work. Perhaps this is why instructors in these schools found the regular textbooks unsatisfactory, for the trend in the past few years has been to do all that is possible to assure understanding to the average child. So much repetition of various ways of developing insight into arithmetical relationships is not necessary for the above-average, and often equally necessary repetition and review are sacrificed.

Asked to give ideas about the causes of weakness in arithmetic, most instructors underlined "Insufficient mastery of basic skills." The percentage holding this view reached as high as 70% in the 5th grade. Therefore, in the description of methods, certain definite features seemed to predominate:

1. Since understanding does not insure memorization, especially in the matter of many little facts that are similar in character, repetition in many forms must be provided if the basic facts are to become the facile tools they should

[7] B. F. Skinner, "The Science of Learning and the Art of Teaching," *Educational Psychology*, ed. by M. R. Loree (New York: Ronald Press, 1959) p. 196–206.

[8] C. Stern, *Discovering Arithmetic* (Boston: Houghton Mifflin Co.)

[9] J. A. McGuinn, *Exercises and Problems in Arithmetic* (West Hartford, Conn., Kingswood School).

be—freeing the mind for later concentration on intricate relationships.

2. Immediate and meaningful goals must be given which will stimulate at each level—in the lower grades games and competitions which insure mastery of the matter and thus give a genuine interest and enthusiasm for the subject in the upper grades.
3. The tendency to rush through work in any manner must be overcome by demanding perfection of presentation in such matters as organization of work, labeling of answers, neatness and accuracy.
4. Since arithmetic is an exact science and entails consistent and tedious work which cannot always be made attractive, real effort and strong work habits must be exacted, and at times, if necessary, reasonable sanctions should be used until the stimulation of success forms the proper motivation.

If the above methods appear traditional, the conclusions should not be drawn that new approaches are not studied and employed. New ideas are constantly being examined, but acceptance is somewhat slow and comes only after careful deliberation, for these procedures have proved successful, as the following discussion on achievement will show.

Achievement

Achievement in E.R.B. elementary grades is measured each spring by the use of the Stanford Achievement Battery of Tests. Standardized tests still seem to be the most satisfactory method of measuring arithmetic achievement, and the Bureau has consistently held to the use of the Stanford tests because of their proved high reliability and validity. One adjustment has had to be made, however, to provide sufficient scope of content material at the various levels. This is more difficult to provide for in arithmetic than in reading, as Gallagher[5] has shown in his recent pamphlet, "The Gifted Child in the Elementary School." The Bureau has solved this problem by placing the tests designated for a certain level down one grade, thus allowing classes who are doing advanced work (as explained earlier in this study under "Curriculum") to perform at the highest possible level. The table below gives an indication of the achievement being reached.

ARITHMETIC ACHIEVEMENT IN E.R.B. INDEPENDENT SCHOOLS—
STANFORD ACHIEVEMENT TEST—MARCH, 1957-

Grade	1	2	3	4	5	6	7	8
Median grade placement of E.R.B. classes	2.4	3.5	4.5	5.5	7.0	8.3	9.9	11.0
Median grade placement—national norms	1.6	2.6	3.6	4.6	5.6	6.6	7.6	8.6

Many factors have contributed to this achievement: the higher level of ability of the pupils, small classes, experienced instructors, strong work habits, and—probably of most importance—the slight acceleration of the arithmetic program. Since only one-third of the schools were attempting this last device, the achievement levels will in all likelihood be raised even higher in the near future when more and more schools follow the recommendations of the N.E.A. in this matter.

Summary

This study of the Arithmetic Program in the Educational Records Bureau Independent Schools has been an attempt to show what is being done in one small portion of the nation's schools for the academically-talented. Undoubtedly much serious effort is being put forth by administrators and instructors in these institutions where experimentation is given enthusiastic support.

The most important findings can be enumerated in a brief review.

Academically talented children, because of their native capabilities, are able to learn more quickly than the average child, so a different program is being provided for them in many of these schools. They can cover the traditional arithmetic course in a shorter period and thus save as much as a year's work, which can be given to the study of higher mathematics. Not only do they learn more quickly, but they are more capable of grasping the meanings in arithmetic. However, they dislike routine and drill work, and unless followed carefully tend to pass over necessary work of reinforcement, and thus acquire habits of carelessness and inaccuracy. Keeping these characteristics in mind, E.R.B. instructors place much emphasis on mastering the fundamental operations early in the arithmetic program. Many are experimenting with the possibility of giving some of this basic work in the kindergarten, for some feel that these children are definitely capable of it.[10] Taking advantage of these early years when children find memory work interesting teachers make basic skills a real part of their learning equipment, so that in the upper grades, when memory work becomes tedious, more time can be given to the development of reasoning powers.

Work habits also play an important role in the program. Personality is developed through systematic training in responsibility, accuracy, and neatness. Such training is somewhat facilitated in these schools, since the desire to succeed in college work is becoming a more and more dominant factor in motivation.

In their efforts to provide for the needs of the academically-talented in arithmetic, instructors of these children are hampered in one important area—the lack of an appropriate series of arithmetic text-books. What is needed is a series that is sufficiently difficult and that takes into consideration the nature of the learning process of above-average children, so that all possible use can be made of their talent.

It is hoped that this study will make some small contribution to the fund of information that is now being gathered from all parts of our country towards the full utilization of the intellectual resources of our youth.

[10] V. C. Simmons, "What's the Matter with Kindergarten?" *The Independent School Bulletin*, May 1958, pp. 13–14.

EDITOR'S NOTE. An I.Q. of 122 places a child at approximately the 90th percentile if one considers that 17 is a fair value for the standard deviation of a range of I.Q. scores. A class having a median I.Q. of 122 is certainly a superior class and should receive an education in quality, kind, and amount accordingly. Too frequently when but a few children of superior ability are found within a class, these few are not given the education they should have. In her summary, Mother Erhart has stated a number of conclusions which are worth rereading and pondering for their implication. Why not put more arithmetic into the kindergarten and grade one? Why not master the drill aspects of computations at an earlier age when pupils seem to delight in this type of learning? Why not devote more time to the mathematical aspects of arithmetic that feature thinking? Why waste a bright child's time dwelling on an idea that he has already discovered and mastered? Are new types of textbooks needed? Why not go back to the older practice of having one or two books for the range of arithmetic instead of one book per school grade? Is it enough to hasten through a common sequence of arithmetic or are there other materials that should be used with the brighter pupils?

Typically in the suburban school one finds children of superior mental ability but one also finds parents who are very much concerned with the learning of their pupils. These parents will support a school with small classes, with a wealth of school materials, and with excellent teachers of experience who are paid perhaps a little more than the driver of the beer truck but not as much as the airline pilot. In these communities the pressure of going to college reaches down into the kindergarten. This can have both a good and a bad effect upon youngsters. It is the school's responsibility in consort with the parents to nurture the child to his best achievement.

University of Maryland Mathematics Project

HELEN L. GARSTENS and M. L. KEEDY, ASSOCIATE DIRECTORS
JOHN R. MAYOR, DIRECTOR
College Park, Maryland

Plan of Operation

IN THE FALL OF 1957 the University of Maryland Mathematics Project (Junior High School) was started. The project was undertaken as a cooperative enterprise of the departments of mathematics, education, psychology, and engineering of the University of Maryland and the four major public school systems in the Washington area—Washington, D. C. public schools, Arlington County (Virginia) public schools, and Montgomery and Prince Georges Counties (Maryland). The project has been guided from its beginning by an Advisory Committee which also includes representatives of the Maryland State Department of Education, the Mathematics Section of the Maryland State Teachers Association, and the United States Office of Education, as well as the cooperating university departments and public school systems.

The project was made possible by a grant from the Carnegie Corporation of New York. The School Mathematics Study Group has assisted with the evaluation of the teaching experiences.

One of the first steps in the development of the project was to invite the superintendents of schools in the four major cooperating school systems to name three to five competent junior high school teachers who would be willing to work on the project for a three-year period. The actual work started with the first session of a weekly seminar attended during the first year by some 25 teachers from the Washington area. In the seminar the teachers heard lectures on modern mathematics and psychology and began to plan their writing of experimental units to be tried out in seventh grade classes during the second year.

A number of units were prepared during the course of the year and tried out by quite a number of teachers working in the seminar program. These units, along with the teachers' comments on the units and their teachability, became the basis for the first draft of the seventh grade course, which was used in the Washington area and a number of other school systems during the school year 1957–58. At all stages in the development of the materials, those junior high school teachers have had a very considerable influence on the choice of topics. For example, one of the very first topics chosen by them for trying out was finite mathematical systems. This choice, the experience in teaching it, and the experience with other related topics emphasizing systems are responsible for the emphasis throughout the text on mathematical systems, one of the characteristic features of the experimental course.

During the first year of the project, it was the intent to associate the writing and teaching of experimental materials with psychological studies. While these studies have been carried on to a limited extent, progress with them has not been as great as had been our original hope. During the second year of the project, major attention of the staff psychologist was given to an evaluation of the teaching of the experimental seventh grade course, and during the third year to the evaluation of the eighth grade course.

During the second year (1958–59) of the project the seventh grade experimental course was taught in approximately 40 classes in the Washington area and by a number of other teachers who attended a 1958 National Science Foundation Summer Institute for Junior High School Teachers

at the University of Maryland. The Washington area teachers attended a weekly seminar at the University. Those teachers away from this area who taught the course were invited back to two conferences during the school year 1958–59 to discuss their teaching experiences and other problems related to the teaching.

This year (1959–60) the experimental seventh grade course is being taught in approximately 75 classes in the Washington area and 25 classes away from Washington. The new eighth grade course is being taught in approximately 30 classes. The seventh grade pupils have available a revised text which was prepared after complete reports had been received on the year's teaching experience.

Language

One of the important problems in the teaching of mathematics is that of language. Not only is mathematics a language itself, but the use of language is necessary to communicate the concepts of mathematics. Apparently the vocabulary and usage now prevalent is an accumulation of several thousand years of effort, and as such contains numerous inconsistencies and unnecessary or confusing terminologies. In writing the UMMaP courses an attempt has been made to improve and streamline vocabulary and usage. A number of items may be mentioned by way of illustration.

It is common practice in traditional mathematics not to distinguish between a symbol and that which the symbol represents. In ordinary language such a distinction is usually clear, but in mathematics it is not. For example, the symbol "cow" would never be confused with the animal which it names, but the symbol "5" is habitually not distinguished from the concept for which it stands. The number five is named by the symbol "5." Such symbols, used to represent numbers, are known as *numerals*. In the UMMaP courses the distinction is made between numbers and numerals, as a first step toward building a rather precise use of language. The motivation for making the distinction is provided in connection with the study of non-decimal numeration. A study of numerals in bases other than ten soon shows that a number may have many different symbols, or names. Those which are based on ten are known as *decimal numerals*, and are the familiar ones. Once the point has been made the way is open to consider and study those properties of numbers which are independent of the manner of naming them, such as the commutative and associative properties of addition and multiplication. A number is even if it is divisible by 2, regardless of how that number is named, and a number is prime if it has exactly two different factors, again regardless of how it is named. That this point of view and the associated manner of speaking have the effect of simplifying and unifying can be seen by considering a traditional kind of problem in arithmetic. Multiplication of common fractions and multiplication of decimal fractions are usually considered as entirely separate problems, whereas in the new view they are simply two techniques for solving a single kind of problem. The problem is to multiply *numbers*. One may choose to use decimal numerals or to use the symbols usually known as common fractions (these are called fractional numerals in the UMMaP course), but the problem to be solved is the same in either case.

The attempt to use language precisely is also carried through in the geometry sections. For example, a line segment is considered to be different from its length. In a circle, any segment with one endpoint at the center of the circle and the other end on the circle is called a *radius*. The length of a radius is a number associated with the radius. Thus in the formula $C=2\pi r$, one substitutes for r the *length* of a radius, but not a radius, because the formula is concerned with numbers rather than line segments. One would not say that the radius of a circle is 6 inches, but that the *length* of a radius is 6 inches. A rather careful distinction is also made between mathematics as an idealization of the physical world and the physical world itself. For example, a line

segment is not to be found in the physical world, for it is perfectly straight and has no thickness. It is an idealization, an abstraction of the nature of actual objects, such as pieces of wire or string, pencil marks, or the edge of a ruler.

An important point which seems to aid in the comprehension of the nature of mathematics is that the meanings of mathematical symbols are a matter of agreement. It is apparently not easy under a conventional curriculum for a student to grasp this fact. In the UMMaP courses the point is made clear by repeated reference and reminder. One kind of activity which is encouraged is the making up of new symbols for numbers. Not only do children find creative enjoyment in doing this, but the point is made clear that at some time the familiar numerals which we use daily were invented in just such a fashion by someone.

Some symbols are introduced in the UMMaP courses which are not used conventionally at the junior high school level. For example the inequality symbols $\neq$, $<$ and $>$, and symbols for exponents. It has also been found possible to effect some economy in symbolism. For example, the division sign $\div$, can be dispensed with early, by explaining that divisions can be written as fractions, such as $8/2$, rather than $8\div 2$. Not only does this provide economy, but it also provides greater facility in performing a number of kinds of calculations.

That mathematics is a language is felt to be a highly important point. Within this language are found sentences. Those which express ideas about numbers are known as *number sentences*. Examples of number sentences are: "$2+1=3$." "$5<7$." "$3\cdot 5<4$." "The number 9 is an even number." All of these sentences express ideas about numbers. Some of them are true and some of them are false, and it is important to realize well that a sentence may be false. The falsity in no way means that the string of symbols is not a sentence. Some number sentences are neither true nor false. In particular sentences containing variables are of this type. Sentences such as $3\cdot x+4=7$ belong in this category. The symbol "x" is called a *variable* because various numerals may be substituted for it. Certain substitutions result in a new sentence which is false, while other substitutions result in a new sentence which is true. Those numbers for which the substitution results in a true sentence are known as *solutions* of the number sentence. It should be noted that this approach is different from that of conventional algebra, where the letter x would be called an *unknown*. That is, it represents some definite but unknown number. In the UMMaP courses the letter x simply serves to mark the place where various substitutions can be made.

Probably the most important kind of number sentence is the equation—a sentence with $=$ for its verb. In conventional arithmetic, where equations are not usually considered as sentences, the meaning of equality is not clear. In the UMMaP courses a specific and simple meaning is given to equality, by agreement. A sentence such as $2+4=5$ states that the symbols $2+4$ and 5 represent the same number. With this as the agreed meaning of such a sentence it is easy to establish that the symbols on either side of the equal sign may be interchanged, or the "sentence may be reversed." This results in greater facility and understanding of this important kind of symbolism.

Mathematical Structure

Much of the traditional content of the seventh grade course in mathematics is also a part of UMMaP's Mathematics for the Junior High School, First Book, but like the material on language is approached from a new point of view. No longer do we need achieve only skill in manipulating with numbers. In addition, the junior high school student is given the opportunity to recognize that the number systems we use are structured like any mathematical system, regardless of the nature of its elements. With this in mind, the Maryland course has, at its core, the concept of a mathematical system. Approximately one-third of the text concerns itself directly with this important idea. The four units which cover

the topic are titled, PROPERTIES OF NATURAL NUMBERS, MATHEMATICAL SYSTEMS, THE NUMBER SYSTEM OF ORDINARY ARITHMETIC, and THE SYSTEM OF INTEGERS UNDER ADDITION. In the unit on Properties of Natural Numbers, the student is introduced informally to the three parts in the structure of a mathematical system, and to the language necessary for communicating about the system. A somewhat more formal treatment is used in the unit on Mathematical Systems, in which systems whose elements are not numbers are considered. With this background the students are ready to set up a mathematical system using the numbers of ordinary arithmetic (whole numbers, fractions) as the elements. Finally, in the unit on the System of Integers, the students are presented with the development of more rigorous definitions of operations and more rigorous proofs for the properties of the system. A more detailed description of the unit on the Properties of Natural Numbers may serve to clarify the student's introduction to this concept.

The three parts in the structure of a mathematical system consist of

1) definition of the elements of the set we wish to use
2) definition of the operation or operations we wish to use
3) proof that the elements and the operations have certain properties associated with them.

In the system of natural numbers, the elements of the set we wish to use are the counting numbers with which the students are thoroughly familiar by the time they reach the seventh grade. The operations we wish to use are addition, multiplication, subtraction and division. In this first presentation, every opportunity is taken to build on the student's background. Intuitive definitions of the operations are acceptable and "proofs" are primarily inductive. Efforts at deductive proofs are most informal in nature.

The title of the unit, the Properties of the Natural Numbers, indicates that the stress here is placed on the third part of the structure of a mathematical system. The properties of closure, commutativity, associativity and distributivity are so fundamental and so basic in all our arithmetic activities, that the student uses them continuously. In many cases the student is unaware that he is applying a fundamental property in his work. He has simply memorized another rule. Sometimes the student does realize that he is using a fundamental property. In almost every case this student thinks he may apply the property because it is inherent in the natural numbers. The student may know that you always get the same answer if you switch the two addends: $3+5$ names the same number as $5+3$. What he doesn't realize is that commutativity is also a property of a mathematical system whose elements are the symmetries of a rectangle, and not natural numbers. The appreciation of the fact that the number systems we use are one kind of mathematical system is a goal of the UMMaP materials.

In this unit the student is given much opportunity to experiment, with the thought that he will come to some conclusions inductively concerning the properties of the system of natural numbers with respect to the four operations. At the conclusion of the unit, the student should be aware of the definition of each property, its purpose, its statement in mathematical symbols, its application in the operations with natural numbers. Since the sum of any two natural numbers is unique and also a natural number, the set of natural numbers is closed under addition. Similarly, the product of any two natural numbers is unique and also a natural number. Hence, the set of natural numbers is closed under multiplication. The student can demonstrate with only one example that the set of natural numbers is not closed under subtraction or division.

Because we may switch the two addends and still get the same sum, we know that addition is commutative in the set of natural numbers. We also get the same product when we interchange the multiplier and

the multiplicand. Hence multiplication is also commutative. However, switching the minuend and subtrahend in subtraction does not yield the same result. Neither does switching the dividend and divisor yield the same quotient. Hence subtraction and division do not have the commutative property.

Addition has the associative property because we may group addends as follows and get the same sum: $3+(5+9)=(3+5)+9$. Three factors may be similarly grouped in a multiplication example, and the product is the same. Therefore multiplication is associative too. Subtraction and division do not have this property in the set of natural numbers.

The properties of closure, commutativity and associativity are used every time the student adds a column of figures in one direction and then checks the sum by adding in the opposite direction. He uses them every time he borrows in subtraction and carries in addition. He enjoys their use when he regroups addends or factors to engage in what he calls "short cuts" in computation. Thus in adding $2+7+8+3$, he may switch and regroup to add $(2+8)+(7+3)$. And in multiplying $2\times 9\times 5$, he may switch and regroup to simplify the example: $(2\times 5)\times 9$.

The distributive property is concerned with both operations, multiplication and addition. Suppose a class is made up of 14 boys and 16 girls, and each plans to bring 10¢ as a contribution to a local charity. What is the anticipated contribution of the class? Some of the students may calculate this by adding the number of boys to the number of girls, and multiplying the sum by 10: $10\times(14\times 16)$. Other students may calculate the amount to be contributed by the girls, the amount to be contributed by the boys, and add the two numbers: $(10\times 14+(10\times 16)$. When the students expect the results to be the same, they are assuming the distributive property for multiplication over addition, for the set of natural numbers.

At the conclusion of the unit, without even mentioning the word "algebra," the students are writing the forms shown on the bottom of the page.

Armed with this language, and these understandings, the student is ready to proceed to a more formal study of a basic structure in mathematics, the mathematical system.

EDITOR'S NOTE. For many years, arithmetic involving percentage and its applications to business and the mensuration of plane and solid figures constituted the larger body of subject matter of mathematics for grades seven and eight. In many school systems grades seven and eight are regarded as the final years of the elementary school. The University of Maryland Mathematics Project is now in its third year and has extended beyond the Washington area. The editor asked for a description of this project so that readers would know what is happening. Any new venture has its supporters and its critics. In support one can certainly say that this new program emphasizes ideas and thinking about and with mathematical structures. The refinement and search for consistency in the use of language is commendable. It might even result in better thinking outside the realm of mathematics. What do we wish for our students in grades seven and eight? Critics of UMMaP have said, "This may be mathematics but it isn't arithmetic." How much and what quality of performance do we want in arithmetic for our boys and girls? Within the limits of computation, can the UMMaP students add, subtract, multiply, and divide with whole numbers and, common and decimal fractions satisfactorily? Can they use these computations in percentage and mensuration? What is satisfactory performance in school? In business and other vocations and trades? In later work in mathematics? Is the UMMaP better for the general citizen than the older percentage and mensuration? We need to consider these things honestly. Can reasonable results in arithmetic be achieved at the end of grade six? Not so many years ago, a number of people argued for a good course in arithmetic in the eleventh or twelfth year because this was considered the common mathematical language of business, householding, and the trades and professions. Is that argument now void? Where do we go from here?

$a+b=b+a$	for the commutative property for addition
$a\times b=b\times a$	for the commutative property for multiplication
$a+(b+c)=(a+b)+c$	for the associative property for addition
$a\cdot(b\cdot c)=(a\cdot b)\cdot c$	for the associative property for multiplication
$a\cdot(b+c)=ab+ac$	for the distributive property of multiplication over addition.

The Impact of the Maryland and Yale Programs*

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I BELIEVE MY REASONS FOR CHOOSING this topic are good ones. First, as they say in the T.V. detective stories, K-12 and beyond is "my beat." As supervisor of mathematics education for the Brentwood Public Schools, it is my primary responsibility to provide the leadership for the development of a comprehensive and modern mathematics program for the young citizens of the Brentwood community. Second, we have, this past year, had some of our students enrolled in the University of Maryland Mathematics Project. Third, our Junior High School will have many more students enrolled in both the UMMaP and Yale SMSG 7th and 8th grade programs.

My plan is to relate to you what I consider to be some of the impacts of these two programs upon a school system such as Brentwood. Specifically I shall mention what has happened to the Jr. H. S. pupils, teachers, and parents; the Sr. H. S. pupils, teachers, and parents; the elementary school pupils, teachers, and parents; the Board of Education, administration, curriculum coordinators, and finally, to the neighboring communities.

Before I swing into the major portion of what happened at Brentwood, I feel I must briefly describe the Brentwood Community. It's like many other Long Island communities which have been engulfed by the mass exodus of New York City residents to the promised lands of suburbia.

When I first came to Brentwood back in 1954 we had an elementary school building which housed both elementary school pupils and the 9th and 10th graders of our growing high school. Today we have 6 elementary schools and at least two more on the way, one Jr. H. S. and another on the way, one Sr. H. S. and another on the way, an evening high school, and adult education program, and finally an evening extension graduate division program from Hofstra College—all of this growth taking place since 1954. Coupled with this phenomenal growth is the fact that the Brentwood Community is considered to be a low-middle income area. Its taxpayers are sorely put to keep up with the rising expenditures necessary to finance their rapidly expanding school system. In spite of the financial barriers and the chaotic growth, the Brentwood citizens, both the old timers and the young blood, are vitally interested in their schools and take great pride in providing a sound educational structure for their children. This is Brentwood in a "nut-shell" or should I say Brentwood in a "bomb-shell."

There remains one more bit of background material before I proceed. By 1957 the mathematics department had completed its first attempts at curriculum revision in Sr. H. S. mathematics. No one in the department felt entirely satisfied with our first product. I applied for an F. S. summer institute scholarship to study at Teachers College, Columbia University in the summer of '57. The summer institute proved to be a rewarding experience for me. I was able to catch a brief glimpse of what our country and its school systems were doing. It was then that I became convinced of the fact that the elementary and Jr. H. S. mathematics programs would need major overhauling if a satisfactory modern high school mathematics program was to be suc-

* Excerpts from a speech given at the meeting of the National Council of Teachers of Mathematics, Ann Arbor, Michigan, August, 1959. For brevity the two programs will be designated as UMMaP and SMSG respectively.

cessfully implemented. In the fall of '57 the mathematics department initiated its second curriculum revision project. In the spring of '58 I went to the New York State Mathematics Teachers convention hoping to hear of what was going on at the University of Maryland. Dr. Keedy, now associate director of UMMaP, informed us of the first results of the UMMaP work and also of their plans for the immediate future. In the summer of '58 Norman Michaels, a member of our Jr. H. S. mathematics staff, and a "died-in-the-wool" social and applied arithmetic teacher, participated in the UMMaP summer institute at the University of Maryland. We figured that if Norm could be sold on UMMaP then he'd be just the man we wanted to spearhead our breakthrough in our Jr. H. S. mathematics program.

I am making a gross understatement when I say that both he and I were very excited with the UMMaP experimental 7th grade curriculum. Believe it or not we were extremely anxious for the '58–'59 school year to get under way. Now as to what happened this past year at Brentwood.

The year started out innocently enough. We had arranged for our four top 7th grade classes to be enrolled in the UMMaP experiment. Norman Michaels was to teach these classes as well as a fifth class using traditional 7th grade materials. Jerry Roberts, a member of our high school staff, taught the 7th grade UMMaP materials to our top 8th grade class. Mrs. Morine, an elementary school teacher, was to teach her class of academically talented 4th, 5th, and 6th graders the 7th grade UMMaP materials. I was to be the free lancer who would fill in any emergency. I must point out that neither Mrs. Morine, Jerry Roberts, or myself were at Maryland during the summer of '58. Jerry and I both had a sufficient background in modern mathematics to be able to teach the UMMaP materials. Mrs. Morine had no special training in mathematics to deal with the UMMaP materials. However, we had no fears as to whether or not she could do a competent job. Not only was she considered to be an exceptional teacher but she also was a former member of a class of academically talented children when she attended elementary school. Norm, Jerry, and I agreed to make ourselves available to her for any help she might require.

We soon met our first problem. Our new secondary schools curriculum coordinator, Dr. Raymond Scheele, and our new Jr. H. S. principal, Dr. Leonard Sachs, expressed genuine concern over the fact that the UMMaP materials were conspicuously lacking in social arithmetic-type problems. As you all know, the social arithmetic-type materials occupy a major portion of a traditional 7th grade mathematics program. They were concerned over the possibility of the school being deluged by parental objections. At the time, this seemed like a definite possibility. We agreed to keep a supply of workbooks on hand just in case we ran into any serious problems. As it turned out, our concern was unnecessary. We were all pleasantly surprised with the overall positive parental response to the UMMaP materials.

At our first departmental meeting, a progress report was made by Norm and Jerry. The reports stated that the children were extremely interested in their work and were doing well. The other members of the department began asking questions about the UMMaP materials. We made some brief comments and provided everyone with copies of the students' text and teachers' commentaries. We agreed to study the materials so that we could discuss them at subsequent meetings.

Later on in the term when the Jr. H. S. P. T. A. held a meeting, Norm Michaels was scheduled to speak to the parents about the UMMaP classes. By this time the parents were prepared to ask a number of questions as well as offer some opinions. We anticipated a lively meeting and we had one. Norm carefully outlined what we were attempting to do and what we hoped the children would learn during the academic year. There were two significant outcomes of the P. T. A. meeting:

(1) the parents had a somewhat better idea of what was happening to their children.
(2) the parents asked the mathematics department to establish an adult education course for parents interested in learning more about UMMaP.

I couldn't have been more delighted since this was exactly what Norm and I hoped would happen. The adult education class was conducted by Norm. He had parents of children from both his classes and Mrs. Morine's class. I visited these classes periodically to see how things were progressing. It was a refreshing experience. These parents were deliberately trying to keep at least one step ahead of their children. It soon became obvious that the parents were having a real battle on their hands trying to keep up with their children. The children were adjusting more easily and rapidly than their parents. However, the parents were genuinely delighted to see the change in their children. They were already beginning to accept the fact that although they were the authorities on matters of family policy, their children were assuming a new intellectual posture in the family scheme.

In the meantime the members of the Jr. and Sr. H. S. mathematics department agreed to go back to school to enroll in courses in modern mathematics. It was indeed obvious to all of us that if they were to be prepared to teach these pupils who would all too soon be enrolled in high school, they, the teachers, would have to be much more than one step ahead of their pupils. So back to school they went in earnest. I'm sure you will pardon me for saying that I am justly proud of the manner in which my colleagues have accepted their professional responsibilities.

It wasn't long before the rumor factories of both the elementary and senior high school pupils began working overtime. This was to be expected since a number of the Jr. H. S. pupils had brothers and sisters in both the lower and upper grades. Naturally these rumblings soon spread to parents and teachers. I don't have the time to describe the many humorous conversations I had with pupils and parents all centering around the fact that whole families were studying mathematics together. We coined a phrase to depict this unique situation, "a family that studies mathematics together grows and adjusts together."

When I noticed that sufficient interest had been exhibited by the elementary school teachers, I met with our district principal, Dr. Eugene Hoyt, who has been guiding the development of the Brentwood School system since 1954. We considered the possibility of initiating an in-service workshop for elementary school teachers. The primary purpose of the workshop would be to acquaint the teachers with some of the modern trends in mathematics education. Dr. Hoyt presented our ideas to the Brentwood Board of Education. We found the Board of Education unusually interested in our proposals. We also found that they were extremely pleased with the positive parental comments they were receiving concerning UMMaP. The Board of Education not only gave us the green light to proceed with our plan, but also agreed to grant in-service credit to those teachers who participated in the course. The workshop lasted 15 sessions of 2 hours per session. We discussed the UMMaP materials and other topics of interest. Some of the important outcomes of this workshop are as follows:

A small core of elementary school teachers acquired a fresh point of view to bring back to their classes. Some of them tried portions of the UMMaP materials on their pupils. Most important of all, they all noticed that their ideas about mathematics teaching were gradually undergoing definite and positive change. They all felt that there was a real need for more articulation between all levels of mathematics teaching. They all felt that they had profited by their experiences and looked forward to a more comprehensive workshop to be conducted the following semester.

The second semester was one of the most exciting and hectic ones our mathematics department has experienced at Brentwood. Hofstra college granted us permission to

conduct a graduate-level methods and content course in modern mathematics education for elementary and Jr. H. S. teachers. Once again I can't take time to discuss with you all of the ideas, methods, and materials we considered. We did have teachers from our own school system, as well as some teachers from other school systems in neighboring communities. The UMMaP materials were considered from time to time throughout the course. I decided to move cautiously in this area since my primary objective was to whet the appetites of the teachers so that they would come to the realization that much more time would have to be devoted to self-improvement. It was my hope that they would enroll in further courses in modern trends in mathematics education. While the Hofstra course was going on, I continued to conduct a series of workshops for our own elementary school teachers. These workshops consisted of 10 days of intensive study of elementary school arithmetic; coupled with the workshop I conducted demonstration classes for teachers. The climax of these workshops was unquestionably a week of demonstration classes, lectures, and evaluations for teachers and parents conducted by an internationally famous mathematician and psychologist. I sincerely hope to be able to report to you at a later date the full story of his visit with us. An important outcome of our workshops was the effort to bring some of the UMMaP materials down to the elementary grades. Our experience to date indicated that pupils in the grades can easily deal with the UMMaP ideas if teachers choose their words and questions carefully and if they accompany their class discussions with the appropriate teaching materials.

The highlight of the Hofstra course for the elementary and Jr. H. S. teachers was a demonstration given by 6 pupils from a class specially created for pupils who had difficulties living up to the accepted social patterns of the average classes. I had worked closely with their teacher, Mr. Morine, who has just been appointed curriculum coordinator of our Jr. H. S. Mr. and Mrs. Morine both participated in our in-service workshop program. Mr. Morine's pupils had developed their own base-six arithmetic, complete with symbols, operations, and tables. They explained their system to the class of teachers through the use of charts and lectures. The teachers were unquestionably impressed by these pupils. The teachers raised one rather interesting question. They wanted to know if students who did not have the special problems of the pupils who gave the demonstration could do the same sort of mathematics studied by Mr. Morine's class.

Norm, Jerry, and Mrs. Morine continued to carry on with the officially enrolled classes of UMMaP. Student and parental interest were still running high. There were no serious problems to report at our departmental meetings. Two of our eighth grade teachers reported on their modest experimental efforts with their slower 8th grade classes. They, too, reported high interest in mathematics and a deeper appreciation of the decimal system of numeration. Here, too, the pupils did a great deal of computation and enjoyed every minute of it. The general mathematics teacher, Monroe Salkin, reported similar results with his classes. His students did a great deal of project work with other systems of numeration, modular arithmetic, scientific notation, factoring, and primes, and historical bibliographic reports of famous mathematicians.

As you can clearly see, things were happening; so much so that I found myself being called upon to address P. T. A. meetings at all levels, local civic association members, as well as a civic association from a neighboring community. All groups wanted to know more about UMMaP and what was going on in the elementary schools to upgrade the arithmetic program.

Just before I left Brentwood for Ann Arbor to participate in this summer's SMSG 7th and 8th grade writing team, I was asked to conduct three three-evening workshops for parents of elementary and high school pupils who wanted to know more about our work. One of the local papers ran a short article whose heading was "Brentwood goes math crazy." Such

was the impact of UMMaP upon the Brentwood community.

At Ann Arbor this summer I learned that the SMSG 7th grade experimental units had met with similar enthusiastic reception throughout the twelve experimental centers. The good news should not have surprised me since both SMSG and UMMaP are similar in spirit as well as content.

This summer's work at Ann Arbor left its impact on me. I met and worked with a group of tireless teachers, supervisors, professors of mathematics education, and mathematicians. Never before in my experience had I met such a group of dedicated, creative, hard-working, and interesting people. I will have many stories both humorous and serious to tell the folks back home.

I would like to suggest the following plan of action for those of you who are contemplating curricular revisions in mathematics education. You have heard it mentioned frequently during this convention that curriculum studies and revisions are best undertaken by a team-approach. The membership of these teams should include mathematicians, professional mathematics educators, supervisors of mathematics, teachers, and most important of all children. These teams should be so structured that instruction, discussion, free exchange of ideas, evaluation, planning, and demonstration classes all go hand-in-hand. I suggest an added dimension, a well informed Board of Education, administration, and citizenry. I also urge you to be well aware of the fact that there is a very real necessity for all of us, whether we be administrators, curriculum coordinators, or teachers, to be thoroughly conversant with all levels of mathematics education through university courses at summer institutes, evening classes, in-service workshops, or visiting-lecturer type arrangements. I would further ask that you give some attention to an experiment being supervised by Dr. John Mayor, the director of UMMaP and chairman of the 7th and 8th grade SMSG project. He and his colleagues are attempting to evaluate a program of elementary school mathematics instruction being conducted by specialists in mathematics education. It may be that the pupil gains in subject matter competency will far outweigh any possible losses incurred by these children not having the same teacher for all subjects.

In conclusion I offer the following warning: The successes of both the UMMaP and SMSG 7th grade experiments are not to be considered as suggestive of curriculum proposals that either project would be willing to grant as final. The achievements to date represent a very good start. I'm sure you'll all admit that these and similar projects are long overdue. I was very pleased to learn that members of this audience as well as others throughout the country have already tried some of these project materials and others at grades below the Jr. H. S. In my conversations with some of these pioneers I found one common thought continually recurring. We are all amazed at the way children readily adjust to the spirit and flavor of these projects.

I shall end this talk by leaving you with the position taken by the mathematician-psychologist who visited Brentwood. After having learned from thousands of children from all over the world he had this to say to me, "Why should we adults be so smug as to assume that we know what children can or cannot learn? Why are we so amazed at the fact that young children can comprehend a substantial amount of genuine mathematics? How well do we educators really know what the capabilities of children are? If we would be teachers of children," he said, "then we must first be prepared to be their students. We must be prepared to learn from children. Above all," he said, "be an interested listener. Learn how to ask provocative questions and tell as little as possible. The children will teach us plenty and in the process become educated themselves."

[*Turn to page 79 for Editor's Note.*]

Tables and Structures

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Experienced teachers know the value of tables of number facts as a means of helping students discover relationships. There is value in directing the student to make his own tables, but often in the interest of saving time, tables already prepared may be placed in his hands. Some of the tables suggested here have been used by accelerated fourth grade classes, others in the seventh grade.

The new approach to teaching mathematics sometimes referred to as the "modern" approach, stresses the fundamental concepts or properties of number, sets and set theory. "The associative, commutative and distributive properties are the 'rules of the game' in manipulating natural numbers."[1] Other rules concerning equality, some of which we use automatically and recognize by intuition are substitution, identity, addition and multiplication of equals to equals. The importance of inverses, inverse operations and inverse elements, closure, and of one-to-one correspondence is also recognized.

As the tables in this article are discussed, an attempt will be made to point out how some of these basic concepts may be introduced to the student.

Some observations and discoveries which might be made by study of the addition Table I might be as follows:

(1) Each counting number is one larger than the immediate preceding one. If "n" represents a counting number, then $n+1$ represents the next consecutive number.

(2) Zero added to any counting number is the number itself. $n+0=n$. This points out that *zero is the identity element in addition.*

[1] Banks, J. Houston, *Elements of Mathematics*, Allyn and Bacon, Inc., 1956.

Table I

Basic Addition Facts

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

(3) Not only does this table show all addition facts for the integers contained in it, but it can also be used to find all the subtraction facts. *Subtraction is the reverse operation to addition.*

The student can be taught to recognize *sets of numbers* which are related by addition and subtraction. For example, if $9+8=17$, then $9=17-8$, and $8=17-9$. To use the table to find the difference between 17 and 8, find the column headed by 8, look for 17 in this column and then look for the number in the extreme left column in the same row as 17. This value is 9.

(4) The table indicates the *commutative law for addition.* This means that when two numbers are combined by addition, the order of combining does not matter. This is expressed by the following: $a+b=b+a$. Each student must realize that $3+4$ is the same as $4+3$ etc. This can be shown as follows: Find the column headed by 3, find the

row indicated by 4 in the left hand column. Follow across this row until you come to the number in the column headed by 3; this value is 7. Then check by finding the column headed by 4 and the row starting with 3, trace across this row to the column headed by 4; this value is also 7.

(5) There are 54 combinations of a binary[2] nature in this table. The student can count these as they are included in the diagonal from upper left to lower right and all the combinations above this diagonal. The fact that the diagonal divides the table into two identical sets of combinations establishes the law of commutativity.

TABLE II
BASIC MULTIPLICATION FACTS

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Some observations and discoveries which may be made by study of the multiplication table above might be as follows:

(1) The product of zero and any number is zero. $n\times 0=0$.

(2) Any number multiplied by one is the number itself. $n\times 1=n$. This points out that *one is the identity element in multiplication.*

(3) Not only does this table show all multiplication facts for the integers contained in it, but it can also be used to find all the division facts. *Division is the inverse operation of multiplication.*

For example: To find the quotient of $35\div 7$, find the number 35 in the table which is in the same row as 7 in the extreme left column. Then look at the number which is the head of this column. This number is 5. Therefore $35\div 7=5$.

(4) This table indicates the *commutative law for multiplication.* This law means that when two numbers are multiplied together, the order of the factors makes no difference. This is shown by the following: $a\times b=b\times a$.

The student can be shown this by checking the product of any two factors such as $5\times 7=7\times 5$. Find the column headed by 5 and the row beginning with 7. Look across this row until you come to the number in the column headed by 5. This number, 35, represents the product of 5×7. Now find the column headed by 7 and the row beginning with 5. Look across this row until you come to the number in the column headed by 7. This number is 35 and represents the product of $7=5$.

(5) The *set of numbers* along the diagonal from upper left to lower right comprises the squares of the numbers. The square of a number is a number raised to the second power. A power indicates that the number is used as a factor as many times as the power indicates. Thus n^2 means "$n\times n$." Students like to know that this is the way scientists write numbers and it is known as *scientific notation.* Table V is an extension of the powers of numbers.

(6) There are 54 possible binary products shown in this table. The products are symmetrical about the diagonal from upper left to lower right which proves the commutativity of multiplication.

The multiplication table suggests "Napier's Rods."[3] Each student can make a set of these for his use on tagboard strips cut half an inch wide and six inches long.

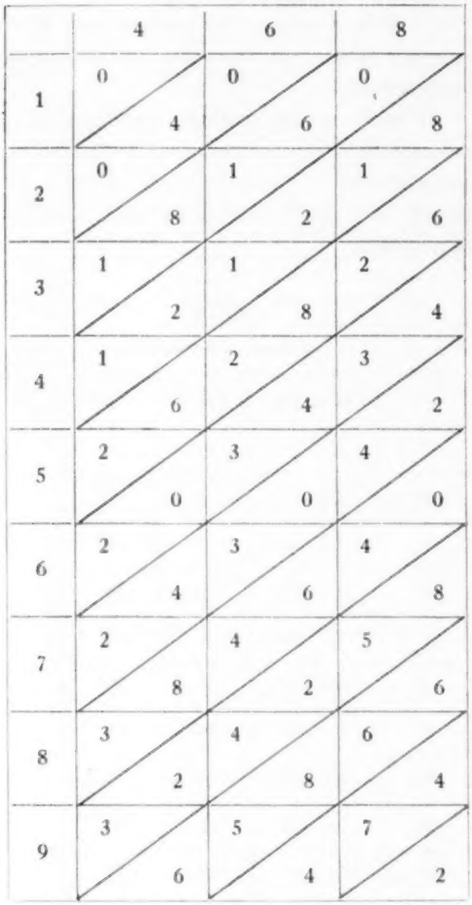
A complete set of rods would include one rod for each integer from zero through 9. An

[2] All operations, addition, multiplication, subtraction, and division are binary operations, i.e. the combining of two numbers at a time.

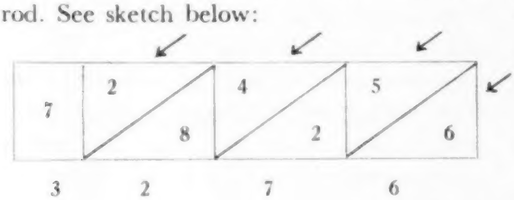
[3] John Napier, Scotch mathematician, 1550–1617. See Sanford, Vera. *A Short History of Mathematics*, Houghton Mifflin Co., 1930.

eleventh rod would correspond to the first column of Table II. By using these rods the student finds he can discover products quickly and accurately.

EXAMPLE: Multiply 7×468



Solution: Select the rods headed by 4, 6, and 8 and place them as indicated. The product is obtained by adding the diagonal columns to the right of 7 on the integer rod. See sketch below:



The product 3,276 can be read easily and quickly by the student.

The author observed an accelerated fourth grade class having the time of their lives using Napier's Rods.

An interesting observation can be made about the diagonal method of arriving at the product as this is related to the "Gelosia" method[4] of multiplying used around the time of Columbus (c. 1478–1500). This method of multiplying makes it possible to record each partial product without doing any addition or transferring. The example below will show how this is done.

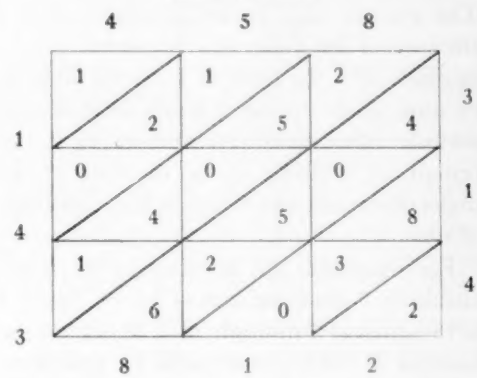
Problem: Multiply 314×458

Solution:

Method used today

```
     4 5 8
    ×3 1 4
   -------
   1 8 3 2
   4 5 8
 1 3 7 4
 ---------
 1 4 3 8 1 2
```

"Gelosia" Method



The product is 143,812

From multiplication Table II can also be studied the sets of products of certain numbers. For example the student may be asked to study the set of products of 9. See Table III. What observations are possible here that may be called unique, i.e. not found in products of other numbers?

(1) The sum of the digits in each product is always nine.

(2) The one's digits are in descending order and the tens digits are in ascending order.

[4] Ibid.

TABLE III

9× 0= 0
9× 1=09
9× 2=18
9× 3=27
9× 4=36
9× 5=45
—
9× 6=54
9× 7=63
9× 8=72
9× 9=81
9×10=90

(3) Each product between 9×1 and 9×9 has a tens digit one less than the factor multiplied times nine.

(4) The products above the line separating the products of 9×5 and 9×6 have the one's digit and the ten's digit in reserved order.

An interesting exploration may be made at this point in discovering *divisibility* by 9. The student may be led to realize that if the sum of the digits of a number is 9 or a multiple of 9, the number is evenly divisible by nine. If, in the other hand, there is a remainder after dividing a number by 9, this remainder is found to be the sum of the digits of the number which is not a multiple of nine.

For example, the number 31761 is divisible by 9 since the sum of 3+1+7+6+1 is 18 which is a multiple of 9. Whereas, the number 21458 is not divisible by nine since the sum of the digits is 2+1+4+5+8 or 20 which is not a multiple of 9, but would have a remainder of 2 if divided by nine.

In teaching the place value of our decimal system of notation, a table of powers of ten is useful.

TABLE IV
POWERS OF TEN

Power of 10	Meaning	Value
10^6=	10×10×10×10×10×10=	1,000,000
10^5=	10×10×10×10×10=	100,000
10^4=	10×10×10×10=	10,000
10^3=	10×10×10=	1,000
10^2=	10×10=	100
10^1=	10=	10

Two observations can be made here that should be helpful:

(1) The power of 10 indicates the number of times 10 is used as a factor.

(2) The power of 10 corresponds to the number of zeros in the value of the number represented.

As soon as the student understands the relation between the power of a number and the number of times it is to be used as a factor, he is ready to make a table of squares of numbers.

TABLE V
SQUARES OF NUMBERS

n^2	Meaning	Value	n^2	Value
1^2	1× 1=	1	11^2	
2^2	2× 2=	4	12^2	
3^2	3× 3=	9	13^2	
4^2	4× 4=	16	14^2	
5^2	5× 5=	25	15^2	
6^2	6× 6=	36	16^2	
7^2	7× 7=	49	17^2	
8^2	8× 8=	64	18^2	
9^2	9× 9=	81	19^2	
10^2	10×10=	100	20^2	

This table may be extended as far as the teacher feels it has value and by doing this the student will get meaningful practice in finding products.

Extension of the use of scientific notation will be found of interest. Making a table of powers of numbers will be valuable if the class wishes to investigate number systems in other bases than base 10. Such a table is suggested in Table VI. Powers of numbers.

Observations that can be made about Table VI.

Table VI

Powers of Numbers

n	n^6	n^5	n^4	n^3	n^2	n^1
1	1	1	1	1	1	1
2	64	32	16	8	4	2
3	729	243	81	27	9	3
4	1024	512	256	64	16	4
5	15625	3125	625	125	25	5
6	46656	7776	1296	216	36	6
7	111649	16807	2401	343	49	7
8	261144	32768	4096	512	64	8
9	531441	59049	6561	729	81	9
10	1,000,000	100,000	10,000	1,000	100	10

(1) Any power of 1 is 1.

(2) All powers of 5 and 6 have the 5 or 6 in the one's place.

(3) All powers of 3, 7, and 9 have 1, 3, 7 or 9 in the one's place.

From Table VI can be constructed the following which shows the place value corresponding to the power of 10 in the decimal system of notation.

Place Name	Power of 10	Value
Million	10^6	1,000,000
100-Thousand	10^5	100,000
10-Thousand	10^4	10,000
Thousand	10^3	1,000
Hundred	10^2	100
Ten	10^1	10
One		1

Compare this table with Table IV.

One way of introducing number systems with bases other than ten is to play a game as follows: Suppose visitors from another planet, say Mars, land on earth. These beings have only one hand with five fingers. They have developed a number system based upon five. How do you suppose he counts? Look at Table VII, Base 5 column.

Table VII

Numbers 1 Through 16 Written in Different Bases

Base 10	Base 9	Base 8	Base 7	Base 6	Base 5	Base 4	Base 3	Base 2
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	10
3	3	3	3	3	3	3	10	11
4	4	4	4	4	4	10	11	100
5	5	5	5	5	10	11	12	101
6	6	6	6	10	11	12	20	110
7	7	7	10	11	12	13	21	111
8	8	10	11	12	13	20	22	1000
9	10	11	12	13	14	21	100	1001
10	11	12	13	14	20	22	101	1010
11	12	13	14	15	12	23	102	1011
12	13	14	15	20	22	30	110	1100
13	14	15	16	21	23	31	111	1101
14	15	16	20	22	24	32	112	1110
15	16	17	21	23	30	33	120	1111
16	17	20	22	24	31	100	121	10000

Here the student can see how to count by fives. Since there is no five in this number system, he must transfer any five to the next column. The place value in the five base would be ones, fives, twenty-fives, 125's, 625's, etc. (See Table VI, row for 5.)

The student should learn to count by 5's before he tries to add or multiply in this base. With a little practice he can easily count as follows, 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, etc. Note: the counting would be said in words as follows: One, two, three, four, one-zero, one-one, one-two, one-three, one-four, two-zero, etc. When writing numbers in bases different

than 10 we can do it as follows: 30_{five}, 16_{seven}, 11011_{two} and 132_{four}.

The student might well prepare a table showing the place value in the base in which he wishes to count. Such a table for base 5 would look like the following:

Power of 5	5^5	5^4	5^3	5^2	5^1	1
Value of Place	3125	625	125	25	5	1

How would the man from Mars add?

Base 5 Addition	Check in Base 10
1 2 4	3 9
3 1	1 6
2 1 0	5 5

Explanation: $1\,2\,4_{five}$ means $(1\times 5^2)+(2\times 5^1)+4$ which is $25+10+4$ or 39. $3\,1_{five}$ means $(3\times 5^1)+1$ which is $15+1$ or 16. The sum $2\,1\,0_{five}$ means $(2\times 5^2)+(1\times 5^1)+0$ which is $50+5+0$ or 55.

Thus the student can see that adding in base 5 is done the same way as in base 10 except that he transfers 5's to the next place value in base 5 instead of transferring 10's as in base 10.

How would the man from Mars multiply?

Base 5 Multiplication	Check in Base 10
1 2 3	3 8
3 4	1 9
1 1 0 2	3 4 2
4 2 4	3 8
1 0 3 4 2	7 2 2

The product 10342 means $(1\times 5^4)+(0\times 5^3)+(3\times 5^2)+(4\times 5^1)+(2\times 1)$ which is $625+0+75+20+2=722$.

This game can be played in any base once the students understand the meaning of transferring as is done in base 10. The more a student works in other bases, the more he appreciates the advantages of base 10, and in addition he is forced to do more thinking to keep his work accurate. The student should be encouraged to check all of his work in base 10.

Table VII shows the numbers 1 through 16 written in different bases (Base 2 through Base 10). Some things that might be discovered by study of this table are suggested as:

(1) The number 1 is the same in all bases.

(2) In even-base numerals, even numbers are easily recognized because the one's place is always even.

(3) In odd base numerals, even numbers can be recognized as the sum of their digits is always even. For example 13_{seven} represents an even number because the sum of 1 and 3 is even (4). Check: 13_{seven} means $(1\times 7^1)+(3\times 1)$ which is $7+3$ or 10 in base ten, an even number.

The tables shown in this article are but a few of many such tables which can be used to advantage in the classroom. True it takes special preparation to duplicate forms that can be used by the student, or the tables would need to be duplicated if one wishes to place them into the hands of the student. However, results will be encouraging whichever method is decided upon. The teacher will find greater interest and fewer students will be found daydreaming or otherwise wasting time. Furthermore, the student will acquire an insight into some of the concepts of the "modern" approach to mathematics.

EDITOR'S NOTE. Mr. Jackson has shown several types of mathematical tables that are useful in showing the organization and structure of arithmetic. Perhaps the chief value in them lies in the study of the structure. Note that such a study, as for example that of multiplication and division, features the "relatedness" of the items. This is in contrast to a prevailing practice of "learning" combinations in isolation. Finally, we do want our students to be able to give products without having to go through a series of preceding products. Thinking, reasoning, and discovery should be characteristic of any approach or method in arithmetic. Since our students are humans with a certain amount of thinking ability we should treat them as such rather than as a "memory machine." These powers of reasoning and the habits of thinking have much greater future potential than mere memory. Tables of sums and products were common in arithmetic books of a century ago. Will this again be true in another decade? But our use of such tables ought to be different from what we believe it was at the earlier period.

Algebra in the Fourth Grade

CYNTHIA PARSONS
Cos Cob, Connecticut

SOME FOURTH GRADERS are really learning algebra. As a matter of fact, so are third, fifth, sixth, and seventh graders. A basic course in algebra has been prepared under the direction of Dr. Robert Davis of Syracuse University. This course consists of a modified form of 9th and 11th grade algebra arranged for presentation to elementary school children. It got its name—The Madison Project—from the Madison School in Syracuse where it was first introduced in 1957.

I am an experimental teacher for the Madison Project, and I teach 2 fourth grade classes. One in Greenwich, Conn., which is labeled average and contains several non-readers; the other in Tarrytown, N. Y., which is considered above average, and contains no non-readers. After only five lessons in algebra, here are some test results.

"ALGEBRA IS FOR THINKERS"

the example	*% who got it correct* Greenwich	Tarrytown
1) $5+\square=25$	96	92
2) $20-\square=19$	100	100
3) $5<\square<7$	57	88
4) $(2\times\square)+2=10$	87	88
5) $5+2+6+8+19+\square=100$	71	58
6) $2+\square+4+7+\triangle=16$	87	75
7) $100-98=\square$	87	92
8) $5\times\square=10$	93	96
9) $(3\times\square)+6=12$	87	68
10) $(\square+8)\times 2=20$	21	57

These 10 sample examples show some amazing results. The median percentage for Greenwich is 87%, and for Tarrytown is 88%. Yet the children in Greenwich have very low Achievement test scores, and those in Tarrytown are quite good. The 6th example—using two unknowns—shows some really sophisticated thinking by these 8 and 9 year old children. They did not have to give all the possible whole number answers for box and triangle, but just substitute one possible set of answers. In several instances, the children did use such combinations as $\frac{1}{2}$ and $2\frac{1}{2}$, and later when we went over the test, the children suggested that the answers should be: $\{(0,3), (\frac{1}{2},2\frac{1}{2}), (1,2), \ldots\}$

The Madison Project material is so arranged that there is the minimum of exposition. Rather, it leads the children into discoveries through a series of questions. The answer to the first is as nearly obvious as possible; the next builds on the answer for the first and so on.

Following is a list of identities compiled in one period in one Weston, Connecticut, sixth grade during an algebra class.

1) $\square-\square=0$
2) $\square\times 1=\square$
3) $\square\times\triangle=\triangle\times\square$
4) $\square\times(\triangle+\bigtriangledown)=(\square\times\triangle)+(\square\times\bigtriangledown)$
5) $(\square+2)\times(\square+2)=(\square\times\square)+(4\times\square)+4$
6) $\square\times(\triangle+5)=(\square\times\triangle)+(\square\times 5)$*
7) $\square\times(\triangle+7)=(\square\times\triangle)+(\square\times 7)$*
8) $(\square+\triangle)\times(\square+\triangle)$
$=(\square\times\square)+[2+(\square\times\triangle)]+(\square\times\triangle)$
9) $\square+\square=2\times\square$
10) $1\times\square=\frac{1}{2}(2\times\square)$

Each of these identities was verified with the substitution of several sets of numbers. Notice how much more sophisticated is the expression of identity 10 than identity 2.

When a probable identity was suggested by one of the students, the teacher would ask for a vote as to how many thought it an identity and how many did not. As soon as the vote was taken, proponents for each case were given opportunities to prove their case.

In my own fourth grade classes, the homeroom teacher is amazed at how much attention is given by each student to the work at hand. It is because they can use their thinking; because a dignity is maintained about

* Pupils found that these are true for any numbers.

the offerings; because they are discussing it with each other, not with an "omniscient" teacher; because they want to try to be right, and every answer is given consideration until it is *proved* unacceptable.

Here is a case in point for sheer ability to think. A boy in the Greenwich class, who has never scored above second grade in an arithmetic achievement test score, thought through this series of open sentences.

$2 \times \square = 4$ *ans. 2*
$2 \times \square = 6$ *ans. 3*
$2 \times \square = 5$ *ans.* $2\frac{1}{2}$

He didn't know how to multiply a fraction by a whole number, but he did *think;* "It must be between 2 and 3, so I think it is 2 and $\frac{1}{2}$."

The present Madison Project Algebra Workbook for the first course is available at cost by writing to: Miss Carol Bowerman, mathematics Dept., Syracuse University, Syracuse, New York. It contains the following: equations, open sentences, inequalities, truth sets, symbol $<$ meaning less than, graphs and tables, identities, and numbers with signs.

"ALGEBRA IS FUN"

A variation of the Japanese game "GO" has been worked out and is called the "Point-Set" Game. At first it is played with the quadrant of the graph using non-negative integers. The children choose themselves into two teams and then alternately, one at a time, give the set of two numbers for the placing of a dot (•) or a circle (0). The point of the game, for them, is to try to surround the dots or circles of the opposing team, and so erase those surrounded—because the winning team is the team with the most dots or circles left on the graph at the end of the period of play.

My fourth graders are particularly fond of a variation of this which we devised. I put a graph on the board and then plot my circles on it. Each of my students keeps an identical graph at his desk, and each one individually plays against me. If one student thinks he has won—that is that he has surrounded 1 or more of my circles—then he shows it to me. Of course, if he hasn't and I can see a move to protect myself from what he has shown me, then I place my next circle there.

I just can't describe with what enthusiasm the children play these games. And as soon as they begin to use the entire graph, requring the use of positive and negative integers, they make the transition easily and quickly. It is constantly amazing how quickly they grasp what it is that comes "before" zero.

At one point, in Greenwich, we were faced with positive 6 minus positive 7. "Impossible," they said.

"If you are thinking of trees or elephants," I said, "then it is impossible."

"Is there really an answer, Miss Parsons?"

"Certainly is."

"Then it must be minus 1, or whatever you call minus 1 in algebra."

"It's negative 1," I said.

"Give us another one, this is fun."

I walked into the Tarrytown class for one of my two lessons a week there, to be greeted with a demand that we play "GO." When I suggested that we play a new game—called the Matrix game*—one bright chap named Gus shouted, "Wow, is algebra ever full of fun! I wonder if trigonometry will be more fun?"

Even if the algebra didn't give the children expanded thinking opportunities, and a facility with numbers which astounds their teachers and parents, it does enough in selling the delight for higher math.

These children go home every Tuesday and Thursday full of their new learning, eager to explain to older brothers and sisters just how to do algebra, and to urge their parents to "catch on" to it.

"ENTHUSIASTIC PARENTS"

I have not yet met with the Tarrytown parents; but was on hand for Open House in Mrs. Coons' fourth grade room one evening.

* Described in "The Madison Project," THE ARITHMETIC TEACHER, December, 1959.

*** "I didn't believe Shelby was really having algebra. And his brother, who is taking high school algebra, said that of course he couldn't be. Shelby was in tears. So I sat down with him and asked him to show me what it was he was calling algebra. Amazing, really amazing. He has taken on a complete new stature. Used to be extremely recessive in front of his two older brothers; now is happy and contributes much."

*** "Tommy has never liked school work. As a matter of fact, although he seems quite bright, has resisted learning and we've often had to come to school to see about his behavior. He loves this algebra. Of course, we didn't think he was really taking algebra until he began to graph equations and to solve equations with 2 unknowns. Is this going to be put into all schools as a regular thing? Too much molly-coddling of the children. This proves they're ready for thinking subjects, doesn't it?"

*** "This algebra is wonderful. First time Helen has done any thinking since she started to school."

*** "Is this the only class getting this algebra? It's a success, isn't it? Certainly is with John. What with the children in the 3rd. grade taking French, and in the 4th grade taking algebra, I have hopes that we (United States) will lead the world again."

*** "Billy bounced into the house last Thursday saying that he had algebra homework. Could hardly wait to do it. And then when I didn't know what an identity was, patiently explained it to me. I really couldn't believe my ears when he said, 'If we do a good job on this, she said she'd give us more homework. Goody!' "

A modicum of formal mathematics is all that is needed in order to teach this algebra. My instruction in mathematics ended in high school.

I met the Madison Project the first week in September of 1959, when I attended a two day workshop in it at Syracuse University. Here it is the middle of November, 1959, and I'm deep in the teaching of 2 fourth grades. The work is provocative, stimulating, instructive, and I predict it will produce skilfull mathematical thinkers.

EDITOR'S NOTE. Yes, algebra in grade 1, 2, 3, 4, 5, 6, 7, 8, 9. But the nature of the algebra develops and becomes more sophisticated. It has value in the development of thinking and seeing mathematical relationships and this in turn can have a very good effect upon arithmetic. The proponents of the types of algebra being described for intermediate grades have no desire to replace the important realm of arithmetic. They too want youngsters to add, subtract, multiply, and divide and to do most of the things now found in a good arithmetic program. But they want more, they want children to learn to discover mathematical relationships and to gain an insight into the nature of mathematical structure. In the elementary school this of course is simple but it is a beginning and often it is amazing what this type of work does to pupils who have been faced with too much computation which is expected to be performed mechanically and with 100% accuracy. Out of the several experimental programs about the country should come a new evaluation of the mathematics program for the elementary school.

The University of Maryland Project

(*Concluded from page 70*)

EDITOR'S NOTE. It is refreshing to note the enthusiasm which Mr. Genise shows in his description of the reawakening of mathematics in the Brentwood schools. No doubt his own enthusiastic personality plus the cooperation of a core of good teachers is a large factor. Good teachers are continually amazed at the kinds of things good students can learn and in which they take delight in learning. Yes, mathematics is "Taking on a new look." And this new look is coming from mathematicians and mathematics teachers more than from curriculum "specialists" and other "educationists." The three best known experimental programs are centered at the universities of Maryland and Illinois, and at Yale University. Each of these programs is honestly seeking a better content and is experimenting with the placement of topics. In evaluating these programs we must consider the final aim for the students. Just because a child can learn something at a given age is not sufficient to dictate that this is the best thing for him to learn at the age. Our traditional content of mathematics for the levels kindergarten through grade twelve has not produced a satisfactory group of thinkers and not even good computers in many cases. Let us be open-minded in viewing newer materials because we must progress. This is an exciting period in mathematics education. Who can predict what our schools will be doing a decade hence? Will our students more generally be delighted with the study of mathematics?

Meeting Individual Differences in Arithmetic

FRANCES FLOURNOY*
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THAT CHILDREN IN THE REGULAR elementary classroom at every grade level vary widely in their interest and achievement in arithmetic is a well-known and accepted fact. It is also recognized that a variety of factors influence pupil interest and achievement in arithmetic and these factors must be taken into consideration in planning ways of helping each individual to progress in arithmetic in accordance with his ability. Everyone recognizes that there is no quick and easy, sure "formula" that can be applied to "clean-up" the problem that every elementary teacher in the regular classroom faces in meeting individual differences. Assuming that each teacher can identify the various factors related to pupil interest and achievement in arithmetic, there still remains the problem of deciding on appropriate variations in learning time needed, in content, in materials, and in methods of teaching. In addition, a type of class organization must be selected which will facilitate the carrying out of variations judged appropriate to meet the varying needs of individuals in the classroom. Presently, the research evidence regarding ways of meeting individual differences in the regular classroom with heterogeneously composed groups of children is inadequate. Teachers and faculty groups might well explore ways of differentiating instruction in arithmetic in the regular classroom situation.**

Suggested Ways of Varying Instruction

Believing that making certain instructional variations is essential when making plans for meeting individual differences, one school faculty[1] has proposed the following types of variations for "try-out" in the regular classroom:

1. Variation in learning time—Examples this are: (a). Allowing the slow learner more time on successive topics and thereby postponing the presentation of some topics until later in the year or next year; (b). Giving shorter assignments to slow learners; (c). Assigning special homework to give the slow learners additional practice; (d). Planning additional arithmetic enrichment activities for the faster workers; and (e). Allowing faster workers to move onto new topics at a faster rate of speed.
2. Content variations—Examples of this are: (a). Adding topics for the fast learner that are not ordinarily found in the course of study as finding median and mode, and learning to count with a base other than 10, while perhaps omitting a few infrequently used topics for the slow learner; (b). Varying the level of difficulty undertaken on any one topic; for example, encouraging faster learners to master the 10's, 11's, and 12's in multiplication and division or having the slower worker do exercises in dividing decimal fractions that involve only whole numbers and

* This paper was presented April 3, 1959 at the Annual Meeting of the National Council of Teachers of Mathematics in Dallas, Texas.

** The term, fast learners, will be used to refer to pupils who are in the upper one-fourth of the class in arithmetic achievement. The term, slow learners, will be used to refer to pupils who are in the lower one-fourth of the class in arithmetic achievement.

[1] Casis School Faculty. *Meeting Individual Differences in Arithmetic* (Directed and Edited by Frances Flournoy and Henry J. Otto), Bureau of Laboratory Schools Publication No. 11, University of Texas Press, Austin, Texas, 1959.

tenths as a divisor rather than spending a great deal of time trying to master division by hundredths and thousandths; (c). Varying the content of practice exercises; (d). Providing rapid learners with more difficult horizontal enrichment involving problem situations in which research is necessary to gather data, as finding the cost of advertising in a local paper. (e). Providing for rapid learners more difficult types of horizontal enrichment which stimulate the mathematical appetite and foster an interest in mathematics as a hobby; and (f). Allowing fast learners to study certain selected topics normally taught in the highest grade. For example, progressing faster with the learning of addition, subtraction, multiplication, and division facts; studying the meaning of per cent in the fourth or fifth grade; and tackling areas of parallelograms and circles in the sixth grade. At the same time a whole school plan for delaying the teaching of a few selected topics for the slower learners might be inaugurated; as delaying the counting by groups of 5's, 2's, and 3's for a grade or more and delaying multiplication and division of decimal fractions until after grade six.

3. Varying teaching methods and materials—Examples of this are: (a). Follow-up reteaching of new skill to slower learners; (b). Frequent review of steps in a process for slow learners; (c). More closely teacher-directed reading of textbook for slow learners; (d). Longer and more frequent use of concrete materials with slow learners; (e). More independent use of textbooks by fast learners; (f). More mental arithmetic exercises for fast learners; (g). Use of the encyclopedia and other such materials by fast learners to investigate certain arithmetic topics; and (h). Differentiated test items with more difficult, unusual, or challenging items for faster learners.

Types of Class Organization

Each of several types of class organization may offer some possibility as an organization in which the teacher can make instructional variations in an effort to meet individual differences. The following possible types of class organization have been proposed:[2]

1. Class-as-a-whole procedure in which the teacher carries all pupils through the arithmetic program for the school year together and gives help and encouragement to individuals as the need and opportunity for doing so are recognized. Efforts at meeting individual differences are usually through helping individual pupils as much as time permits during the arithmetic period, occasional variations in practice content, occasional extra arithmetic activities of a different kind for some pupils, and whatever enrichment activities the teacher might encourage rapid learners to pursue under their own steam.
2. Combination of whole class and small group organization. Each new topic is introduced to the class-as-a-whole. The class is later grouped so that the teacher may reteach when necessary, use varied materials, provide practice on different levels of difficulty, and provide different learning activities. When another new topic is undertaken, the class again works as a whole.
3. Grouping the class in two or more groups according to arithmetic achievement.
   (a) One plan might be to keep the class together on the same area of content but teach each group separately and vary the learning activities, visual aids, and practice as needed. The teacher alternates time spent with each group.
   (b) Another plan might be to make effort to keep the class together as

[2] *Ibid.*

successive topics are studied. At the beginning of the school year, the class is divided into two or more subgroups; each group moves forward at its own rate according to the logical sequence of arithmetic topics.

4. Completely individualized instruction in which each pupil proceeds from one topic to another at his own rate. Each child would be using some self-teaching materials at his level of achievement. It was recognized by these teachers that this plan is limited by the great possibility that pupils might perform too much on a mechanical level only, with too little mathematical understanding and social appreciation. It appears the motivation problem would be especially acute for slow learners.

Though each of these types of class organization offers some opportunity for making instructional variations, it is judged that types 2 and 3 probably offer more opportunities to differentiate arithmetic instruction effectively.

An In-Service Project to Meet Individual Differences

A group of 36 teachers[3] has attempted to explore the advantages and disadvantages of using certain teaching variations in different types of class organization. In the beginning all of these teachers were using a strict class-as-a-whole plan for teaching arithmetic. They expressed the feeling that they were giving considerable individual attention to the slower learners but were not sufficiently challenging the more able learners.

In order to explore possibilities for making instructional variations within each type of class organization, the teachers voluntarily agree to use certain types of class organization and to keep a brief record of daily plans. Seven teachers agreed to use the combination plan of whole class followed by small groups as needed (Type 2). Eight teachers decided to try out a grouping-by-achievement plan in which the teacher would alternate her time between two groups. The middle-average to above-average pupils in achievement were placed in one group and the low-average to below-average pupils in achievement were placed in another group. Three of these eight teachers decided to let each group move from one topic to another at its own rate of speed rather than trying to keep the whole class working on the same general topic (Type 3b). Five of the eight teachers who were grouping by achievement preferred to keep both groups on the same general topic though each group would be taught separately while using some variation in methods and materials and variation in practice as well as enrichment activities (Type 3a). The remainder, 21 of the 36 teachers, preferred to try to meet the needs of children in a whole class organization.

For this exploratory investigation the following plans were agreed upon for all types of class organization:

1. No teacher would use any more time for arithmetic than that teacher normally does in a whole class situation.
2. The same basic core of topics would be introduced to all pupils using the textbook as a guide.
3. Attempts would be made to vary practice exercises to suit the needs of pupils.
4. Content variations would be in depth and level of difficulty rather than by accelerating the program for the fast learners to include topics normally taught in the next grade and delaying until the next grade the teaching of certain topics to slow learners. The latter is not being discounted as a possibility but it was judged best to explore first other ways of varying content.
5. All teachers in grades 3 through 6 would make effort to take full advantage of suggestions for slow and fast learners in the teacher's edition of the arithmetic textbook. Though there are general suggestions, there is no special

[3] Thirty-six teachers in the public schools of Calhoun County, Texas.

help of this kind in the first and second grade worktext materials which were being used at the first and second grade levels.

Through classroom visitation, conferences with teachers, and reading of written records, the following observations were made during this exploratory investigation:

1. Teachers using a grouping plan or a combination whole class and small group plan seemed to be more alert to the special needs of both the slow and the fast learners. As one teacher said, "I discover children's needs better now."
2. Teachers using a strict class-as-a-whole plan appeared to be generally more reluctant to try out variations in content or learning activities or, at least, less frequently seemed to do so.
3. Just the idea of whether to group or not to group seemed to be the major question in the beginning. Later more of the teachers saw class organization as a procedure which might enable them to make more appropriate variations to meet the varying needs of children. They came to realize that just the act of teaching by groups is no answer to the problem of meeting individual differences.
4. These teachers did not feel a necessity to use the same plan of class organization all the time. Several teachers using a certain plan in the beginning began using two plans for class organization. The plan of organization varied with the topic or stage of development of the topic. Some teachers changed to an entirely different plan. Since this was an exploratory project rather than a controlled experiment, teachers were encouraged to make adjustments in class organization and to use the organization that seemed to serve best in the teaching situation at any one time.

   One teacher stated, "In the beginning I had seven Latin-American children who had already spent one year in the first grade, they did not need to go through again some of the very simple things that the beginners still had to learn. In addition, I had four children who were very slow and needed to be grouped alone in order that the other members of the class not be held back to their rate of progress. The middle group caught up with the group of repeaters and will soon be ahead of them. Now I usually introduce everything new to the whole class and also give time to the four slow learners in a group to themselves. Grouping helped a great deal in the beginning."

   Another teacher said, "I am using two groups and each is moving at its own rate. At the beginning of school, no one was progressing when I was trying to teach the whole class the same thing at the same time and in the same way. I am satisfied that all are progressing now. However, on some topics, especially when teaching the different measures, I bring the whole class together for the introduction."

   Three of fifteen teachers, who began by using some type of grouping plan, changed back to whole class organization. These teachers judged, that in their particular classes, plans involving grouping were not especially advantageous because the range of achievement was not wide in their classes.
5. Of the 21 teachers who were using the whole class plan of organization, four expressed the belief that they were not meeting individual needs as well as they might through some work with achievement groups. At various times during the year whenever a special need was recognized, these teachers used a two-group plan based on achievement on a particular topic or skill.

   The other 17 teachers expressed the belief that they were doing as well in meeting individual differences in arithmetic as could be done with other types of class organization.

6. Generally, all of the teachers using some type of class organization which varied from strict class-as-a-whole procedure made occasional time variations and variations in methods and materials. Content variations were made infrequently. Evidence of some kind of content variation was more often observed in classes which were not using a strict class-as-a-whole procedure.
7. In general, these teachers did not seem to use to full advantage the extensions suggested in the teacher's manual for the fast, average, and slow learners. They frequently moved on to the next page or topic in the textbook without further extension of the previous topic. However, this is a new textbook series with this group of teachers and it is partly a problem of getting acquainted. In addition, a plan for varying arithmetic activities and content for children on different levels of ability is a new effort for many teachers. One teacher who was using the combination whole class and small group plan said, "I do use extensions, but I am going so slowly; everybody else is so far ahead of me. However, the children are interested and I am more watchful for ideas to enrich arithmetic." There were several teachers for whom the manual suggestions seemed to serve as a stimulus to make a greater effort to meet individual differences.

Arithmetic Textbook Series Aid the Teacher

What procedures are used in arithmetic textbook series for aiding the teacher in meeting individual differences? Six arithmetic textbook series were examined. Except for the diagnostic testing program, two of the series present suggestions about and also activities for use in meeting individual differences entirely within the teachers' edition of the text.

Another series presents the diagnostic testing program, pages of extra practice, and a few enrichment pages within the children's text. In the main, however, those preparing this third series express the belief that suggestions within the textbook do not deserve to be called "a program for meeting individual differences." Therefore, the major part of this series program for meeting individual differences is included rather consistently in the teachers' edition. Suggestions are specifically designated for slow, average, and fast learners.

A fourth series has occasional suggestions for slow and fast learners included in the teachers' edition. In the textbook for this fourth series there are scattered pages of recreational arithmetic and pages of questions and word problems on a little more difficult level.

A fifth series includes most of its program for meeting individual differences within the children's textbook. This includes tests for diagnosing difficulties, extra practice, and occasional enrichment exercises. Additional enrichment suggestions are given in the teachers' manual for this fifth series. Optional pages in the textbook which may be left out for the slow learner are noted in the teachers' edition. Finally, a sixth series includes practically all of its program for meeting individual differences within the children's text but included fewer suggestions than found in other series.

The inclusion of a more complete and consistent program for aiding the teacher in meeting individual differences is a generally new effort on the part of those who prepare textbook series. Many teachers are still in the process of learning how to use these suggestions to the best advantage. Faculty groups need to study the program for meeting individual differences in arithmetic as this is organized in the particular arithmetic series being used for there are considerable differences in these textbook plans. In order to use effectively the arithmetic series program for meeting individual differences, each teacher must be thoroughly acquainted with the plan, recognize the particular value of each type of exercise for the pupil, and be aware of ways in which these suggestions

may be used for certain pupils at the appropriate time within the regular arithmetic period.

The following techniques and types of activities are being included in textbook series to aid the teacher in meeting individual differences. Certain techniques are used rather consistently in some series while others receive only occasional use.

(a) Testing: Readiness, diagnostic, and supplementary.

(b) Varied practice and review: Self-help pages, extra practice, easier practice exercises, sets of harder practice.

(c) Horizontal enrichment: Recreational arithmetic and historical information. Included are games, puzzles, number oddities, references to number stories.

(d) Varied exercises to provide horizontal extension on skills learned as:

Make up examples to specification (a four place dividend; and a two place divisor that the quotient has three places). Write your height and weight using Roman numerals.

Write a summary about how to carry in the addition of two numbers.

Write as a multiplication example (24+24+12+12).

What is the relationship between a given rectangle and a new rectangle in which the length is doubled? What is the relationship if both the length and width are doubled?

(e) Extended social applications as:

List types of situations that require use of division.

Make graphs to show hourly variations in outdoor temperatures.

Study time tables; compute differences in time and cost of trip when made by train, plane, and bus.

Clip advertisements of rugs. Use these prices and measurements of rugs at home to write problems about cost of putting in rugs or replacing old rugs with new ones.

(f) Projects: Draw a community map to scale; put up a bulletin board on How We Use Fractions; give oral reports on the history of measures; make a time line to show ages of certain well known buildings.

(g) Constructing visual aids; Make fraction kits for class use; make a chart to show the meaning of proper and improper fractions.

(h) Suggestions for varying teaching techniques and materials: For the faster learners, more and more difficult mental arithmetic exercises, more research exercises and more independent problem construction, and the use of a variety of ways of proving. For the slow learners longer use of materials, more teacher guided oral work, dramatizations, omitting certain difficult exercises for the slow learner, and reading problem tests aloud for these children.

Conclusions

Regarding the problem of meeting individual differences in arithmetic, the following conclusions are made:

1. Teachers seem generally to agree that the task of meeting individual differences in the regular classroom makes necessary some teaching variations. Certain variations in learning time, content and teaching methods and materials seem to offer possibility. At this time we have no significant published research in the field of arithmetic which aids a school faculty or a teacher in the decision as to what variations to make, when, how, and how much. Many teachers have a fine attitude toward the problem of meeting individual differences, however, the task of selecting varied activities and getting these ideas before the children at the right time continues to be difficult for teachers to handle.

2. There appears to be a close relationship between the teacher's willingness to vary from a strict class-as-a-whole teaching situation and the use of varied

content and procedures which might aid in meeting individual differences. Less variation in an attempt to provide the extra help needed by the slow learners and the extra challenge needed by the fast learners has been observed in classrooms using a strict class-as-a-whole teaching organization. Opportunities for meeting individual differences in a class-as-a-whole situation might well receive some careful study since the majority of the teachers appear to teach arithmetic in a class-as-a-whole organization.

3. Presently a flexible plan for the use of some whole class teaching in arithmetic and some teaching by small groups seems to be favored by the teachers desiring to vary the class organization in an effort to facilitate the use of variations for meeting individual differences. Teachers willing to try out varied class organization learn to recognize situations or activities in which the class as a whole can work well together and other times when the topic or activity seems best carried out through the use of small groups.
4. Textbook series are offering a more complete program to aid the teacher in meeting individual differences. Faculty groups would profit from study and discussion of the textbook series program in providing for individual differences. A full acquaintance with the program is necessary in order to use it with understanding and with ease. In addition, a thorough acquaintance with the textbook program is necessary if the teacher is to be in a position to supplement the program where necessary to do so in meeting the needs of a particular group of pupils.

   It seems that teacher preparation of exercises that are different and on a more challenging, provocative level is very difficult for most teachers. Textbook series can help much by including this type of exercise as explained in type (d) above.
5. This exploratory investigation of the problem suggests that those teachers who are willing to try out some variations or differentiations are making some progress with this problem of meeting individual differences in arithmetic.

EDITOR'S NOTE. Yes, there is no easy "formula" that will solve the difficulties encountered in teaching a class of thirty pupils who represent a wide range of differences in ability, in background, in will to learn, and in all the other aspects of human kind and quality. But to be conscious of the situation and to try to do something about it is a good starting point. Dr. Flournoy has described the operation of the more common standard practices in meeting individual differences. Flexibility of arrangement is usually more satisfactory than establishing and maintaining one special procedure. Teachers also have individual differences and the procedure that works well for one may not prove to be especially valuable to another. Always we hold as our unattainable aim that each student should learn at his optimum rate. We must be content in the admission that we cannot teach all pupils as well as we should like and that progress will probably be less than we desire in many cases. That is the way human beings react with one another. But let us always keep trying to do the best job we can with each individual pupil.

A Suggested Correction

On page 217 of THE ARITHMETIC TEACHER for October, 1959, we note the following misstatements: 8 hr. 45 min. = 8 hr. 50 min.; 3 23/24 = 4; 5 = 5 5/12; 2 ft. 10 in. = 3 ft.; and others.

We recognize the purpose of the author of the article, but his egregious misuse of the equality sign should not go uncorrected. A permissible notation might be the following:

$$\left.\begin{array}{r}9\text{ ft.}\\ -2\text{ ft. }10\text{ in.}\\ \hline\end{array}\right\} \leftrightarrow \left\{\begin{array}{r}9\text{ ft. }2\text{ in.}\\ -3\text{ ft.}\\ \hline\end{array}\right.,$$

where the symbol "↔" means "is equivalent to."

Yours truly,

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Using Equations with the Number System

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Equations can be used to deepen understanding of our number system when they are constructed in such a manner as to channel thinking into a left-to-right computation. This is often performed in "Mental Computation" utilizing *groups* to arrive at sums, remainders, etc; e.g., 49 thought of as 40 and 9, *not* as 4 tens 9 ones; 236 thought of as 200 and 36, or 230 and 6, or 200 and 30 and 6, *not* as 2 hundreds, 3 tens, 6 ones. This types of computation differs from the kind of thinking done when standard forms of algorisms are used in operations with whole numbers and fractions.

I: Relating to Ten

Use of Associative Principle in Addition and the Decimal Idea.

A. *Addition of 2 one-place digits whose sum is a 2 place numeral*

1a.	$8+3=10+N$	b.	$8+3=N+1$
	$8+4=10+N$		$8+4=N+2$
	$8+5=10+N$ etc.		$8+5=N+3$ etc.
c.	$8+6=10+N$		
	$8+9=10+N$		
	$8+7=10+N$		

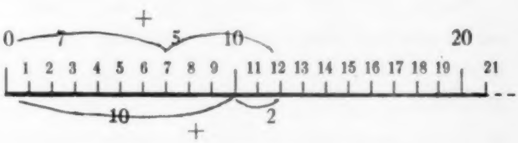
2a.	$9+4=10+N$	b.	$7+4=N+1$
	$7+5=10+N$		$9+6=N+5$
	$9+2=10+N$		$8+3=N+1$
	$8+6=10+N$ etc.		$7+9=N+6$ etc.

The above equations can be used as practice exercise material after development of the second decade addition facts. If all facts whose first addend is 8 are presented and developed with the use of manipulative structured materials (relating to 10), then the first group of equations 1 a, b, c might be given to the abler group of students who grasped the idea readily and need little or no reference to manipulative materials for these facts.

The second group of equations, 2 a, b, is a mixed practice exercise that includes 8, 9 or 7 as the first addend. This can be used for practice after development of all the second decade facts: $9+2$, $9+3$, through $9+9$; $8+3$, $8+4$ through $8+9$; $7+4$, $7+5$ through $7+9$.

Application of above to a Number Line (for a higher grade level). Show the equation on the number line.

$$7+5=10+N$$



B. *Higher Decade Additions; Higher Hundred Additions* (relating to 10)

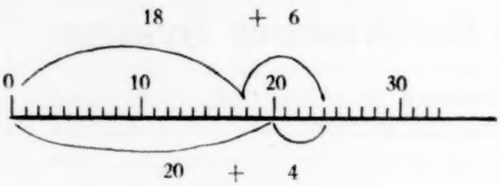
a.	$18+6=20+N$	b.	$37+6=N+3$
	$39+4=40+N$		$49+5=N+4$
	$98+5=100+N$		$96+5=N+1$
	$106+9=100+N$		$118+4=N+2$
	$264+8=270+N$		$307+7=N+4$
	etc.		etc.

Note: Use of manipulative materials such as a Tens-Ten Frame (100 beads —10 beads on a row, 10 rows) aids in the development of this type of thinking for higher decade additions. For larger numbers, solutions should be arrived at by what is known of the sequence of numbers and the type of thinking for higher decade additions.

Application to a Number Line

(A higher developmental level: for some students in the same grade as above; for others in higher grade.)

Complete this equation. Prove it on a number line.



$$18+6=20+N$$
$$18+6=N$$

C. *Addition of Two Two-Place Numerals and Larger Numerals* (Using associative principle, 10 and multiples of 10, 100 and multiples of 100)

1a. $24+17=30+N$ b. $267+\ 34=290+N$
$24+17=34+N$ $\quad 267+\ 34=297+N$

$36+29=N+15$ $\quad 347+234=500+N$
$36+29=N+\ 9$ $\quad 347+234=547+N$

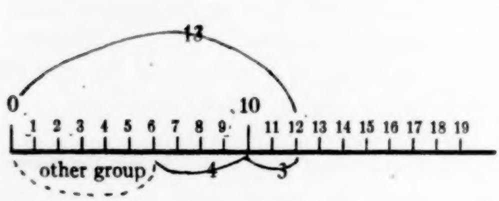
These two-digit numbers may be placed on a number line and the addition represented in the equation will become more graphic.

D. *Subtraction: Second Decade Facts.* (The minuend a 2 place numeral, the subtrahend and remainder a one place numeral.)

1a. Balancing Equations
$11-2=11-1-N$
$11-3=11-1-N$
$11-4=11-1-N$ etc.
b. $11-4=11-1-N$
$11-8=11-1-N$
$11-2=11-1-N$ etc.

The same developmental procedures apply to subtraction facts as to addition facts.

3. Application to a Number Line



$13-7=6$

$$13-7=13-3-N$$
$$13-7-N$$

E. *Higher Decade Subtractions; Higher Hundred Subtractions*

1a. $31-3=31-1-N$
$31-3=N$
b. $147-9=147-7-N$
$147-9=N$

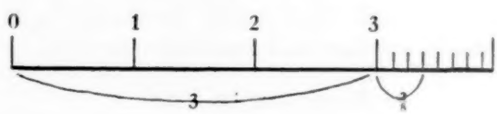
These decade subtractions as well as those using two or more digits can be represented with various equations and can also be shown on a number line.

II. Fractions

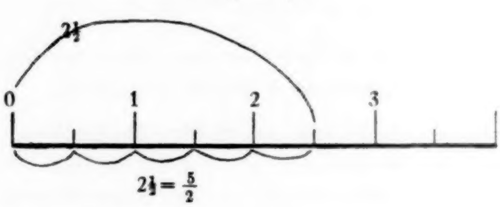
A. *Equivalent Fractions*

Fill in the missing numerals. *Construct* a number line* to show these are equivalent.

1. $3\frac{3}{8}=N+\frac{3}{8}$

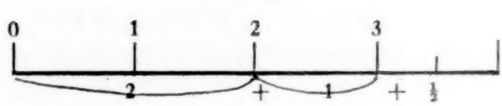


2. $2\frac{1}{2}=N/2$

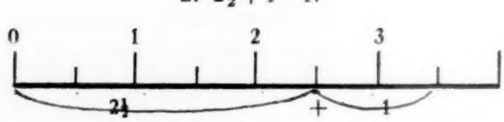


B. *Addition of Mixed Numbers with Unit Fractions*

1. $2+1\frac{1}{2}=N$



2. $2\frac{1}{2}+1=N$



Similar developments may be constructed using mixed numbers in varying degrees of

* When children construct their own number lines in fraction work, it is a more effective learning aid.

difficulty depending upon placement in the equation and the relationships of the fractional parts of the numbers.

C. *Multiplication of Fractions*

Use of Distributive Principle

1. Unit Fraction	$3\times 2\frac{1}{4} = 6+N$
	$3\times(2+\frac{1}{4}) = 6+N$
2. Non-unit Fraction	$2\times 4\frac{2}{3} = 8+N$
	$2\times(4+\frac{2}{3}) = 8+N$
3. Unit Fraction	$2\frac{1}{2}\times 2 = 4+N$
4. Non-unit Fraction	$3\frac{3}{4}\times 8 = 24+N$

Distributive and Commutative Principles

5. $8\frac{1}{2}\times 2\frac{1}{4} = 18+N$
6. $2\frac{1}{4}\times 8\frac{1}{2} = 17+N$

An attempt has been made in the above presentation to show the continued development and application of a few of the basic principles of our number system to computation of various sizes of whole numbers and of fractions through the use of equations and number lines.

No attempt has been made to suggest grade levels for the introduction of these and other equations and of number lines, since children at various age levels and grades differ greatly in ability and forms of equations differ in difficulty.

EDITOR'S NOTE. The equation form of expressing number combinations and with the symbol N is used by a number of teachers because they find that it is more encouraging for understanding and that pupils who learn in this fashion depend less upon pure memory. The sequences presented in this article are suggestive. The fuller sets of materials which would be used by teachers with pupils can be supplied once the idea is accepted. Many schools are experimenting with different approaches to the basic facts and principles of arithmetic. The main aim is still one of understanding plus the ability to think and to perform.

The Equation Method of Teaching Percentage

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THE EQUATION IS THE BASIS of all problems concerning arithmetic, mathematics, chemistry, physics, and engineering. A problem is an incomplete equation. When you get the answer, the equation is complete. Perhaps we should rewrite our elementary arithmetic books and teach children from the beginning to solve equations. In our present method, we let our children run into equations like "hitting a stone wall" in Algebra. Why not set up a system of problem solving in the elementary school which is basic to all problems? Ratios will be easier in Chemistry and Physics when children are taught to solve equations from the very beginning.

Probably most elementary teachers are now ready to quit reading this article as they feel the equation is too difficult for their children. The easiest combination in the second grade is an equation. The teachers do not think of it as an equation, so why should the children? The equation sign is the most important thing we have in math and science.

Those of us who have tried the equational method in the teaching of percentages know that it is easier than the three case method. Children understand percentage problems. They can solve all three cases without deciding which case they are solving by some certain process which they might confuse. They would be solving incomplete equations the same way they did in third and fourth grade.

In the equational method, there is only one "case" to solve all percentage problems and that is to solve the incomplete multiplication equation. Let's look at some simple fourth grade incomplete multiplication equations:

$$3 \times 4 = \_\_\_\_$$

$$3 \times \_\_ = 12$$

$$12 = 3 \times \_\_\_\_$$

Rule: Multiply if you can. If you cannot multiply, you divide one factor into the product. Children must know:

$$\text{factor} \times \text{factor} = \text{product}$$

or

$$\text{product} = \text{factor} \times \text{factor}$$

When you teach percentage, you start out with an English lesson. "Is" is the most important word in the sentence. It is the verb. You cannot have a complete sentence or question without a verb. "Is" in percentage always means "is equal to" and can be read that way. Mathematically "is" or "is equal to" is your equation sign. After all, that is is what the English language is telling us. It is quite easy for children to understand "of" when used with fractions, decimals, or percentage. Then "of" means "times" or is the multiplication sign. "What" or "what number" is your unknown number which is written as a blank space in the incomplete equation. "Find" when used in percentage begins a command. This type of problem can easily be changed by the children to a question which is your problem or incomplete equation.

Find 25% of 32.
What is 25% of 32?
What number is equal to 25% of 32?

Now we translate from English to mathematics putting the math terms in the exact order they are in the question.

What number is equal to 25% of 32?

$$\_\_\_\_ \quad = \quad 25\% \times 32$$

$$\_\_ = 25\% \times 32$$

If the student does not know how to solve this incomplete equation, we write a simpler one he can solve:

$$\_\_\_\_ = 25\% \times 32$$

$$\_\_\_\_ = 3 \times 4$$

We still use the same rule we used in the third and fourth grade: Multiply if you can. If you cannot multiply, divide the factor into the product. This problem they can multiply and complete the equation.

8 is what % of 32?
8 *is equal to what number* % of 32?

$8 \quad = \quad \_\_\_\_ \quad \% \times 32$

$8 = \_\_\_\_\% \times 32$

Here we divide the factor into the product. If the student has trouble, ask him to write a simpler incomplete equation in the same order:

$$8 = \_\_\_\_\% \times 32$$

$$8 = \_\_\_\_ \times 2$$

He solves the percentage equation the same way he solves his simpler equation.

This problem tells the student to multiply a blank number by another number. The factor is always nearer the multiplication sign. The product stands "alone" on one side of the equation sign. There are two factors and only one product. In this problem, only one factor is known so he cannot multiply. You never multiply the product, but you divide into the product.

8 is 25% of what number?
8 *is equal to* 25% of *what number?*

$8 \quad = \quad 25\% \times \quad \_\_\_\_$

$8 = 25\% \times \_\_\_\_$

$12 = 3 \times \_\_\_\_$

We only use the simpler equation at the beginning of percentage. Children quickly follow their rule and make no mistakes in trying to decide which case they are solving. Percentage becomes another form of decimal work based on the incomplete equation.

We still have the complex question in percentage. If children haven't studied the complex sentence, then the math teacher is going to have to teach English again. In the complex sentence or question, we have one main sentence and a dependent clause. The clause depends on the main sentence for its meaning and is nothing when written by itself. In the complex percentage questions, we have two questions, one depending on the other for its meaning. Really, both questions have the same unknown number. Your dependent question really has two unknown numbers and means nothing by itself. You cannot solve a multiplication equation when you know only one number.

If 50□ of a number is 20, what is 10% of the number?

"If" is your conjunction and tells that you have two incomplete equations of different values.

If: 50% of a number is 20

$50\% \times \_\_\_\_ = 20$

What is 10% of the number?

$\_\_\_\_ = 10\% \times \_\_\_\_$

Since one number is common to both equations, let us use the capital N.

$$50\% \times N = 20$$

$$\_\_\_\_ = 10\% \times N$$

Your question says that your N's are the same number. You can solve the first equation to find "N." Substitute that number in the second equation in place of "N" and solve.

This method makes the formula method of solving interest problems easier. Your formula is an equation. Your children have already worked with equations. When we have three factors, no matter what kind of a problem it is, the rule is: Multiply first, if problem is not complete, you divide into the product. In all interest problems, the amount of interest in dollars and cents is your product.

I have never taught second or third grade arithmetic. Tests going through the principal's office show that children have more trouble with written problems than straight fundamentals. Is it because they cannot restate a problem into a simpler question? Primary teachers can make a similar rule for addition and subtraction. Perhaps written problems in primary grades can be asked or reworded so they become a simple question, which is an incomplete equation requiring addition or subtraction. I believe the addition equation to be basic from which subtraction should develop. Perhaps the primary children need help in transferring from English to math terms into the forms of very simple incomplete addition equations. From my work with children using the multiplication equation, I believe the addition equation can be used with better results in second and third grade.

This is what I mean when I say addition should be the basic form:

$$3+4=\_\_\_$$

$$4+\_\_=7$$

$$7=3+\_\_\_$$

Primary children can solve incomplete equations with very little help which will start their foundation for all math and science equations.

EDITOR'S NOTE. Mr. Kessler who is a school principal has long been interested in mathematics. He is interested in having children translate from a sentence to a mathematical statement or equation. The simple declarative sentence is so like an equation that no difficulty should be encountered and yet many pupils find this to be so. To solve the three "cases" of percentage is not so difficult when these are not clothed in a descriptive situation as in an applied problem. Most pupils are troubled in identifying the essence of a problem. Mr. Kessler avoids the algebraic symbolism often used with percentage, i.e. $b \times r = P$. There are several approaches to percentage all of which use the method of the equation. Why not, as Mr. Kessler suggests, do more with the equation at all grade levels? We use it in grade two with a question mark or blank to represent the unknown. Why not introduce the symbol N at an early stage and think with the relationships expressed in terms of N? e.g. $4+N=12$; $20-N=12$; $2 \times N=8$; $\frac{1}{2}N=5$; $N/4=5$; $2N+3N=30$; etc. Teachers who have tried this approach have marveled at the thinking it engenders and surprised at the ability shown by many pupils.

Food for Thought

One of the objectives of education is to help children appreciate our civilization through an understanding of its culture. It would seem necessary, then, that attention be given to this all important phase of mathematics. Should not the teacher then be expected to help the child develop an understanding of the structure of the decimal number system plus an appreciation of its simplicity and efficiency.

As an example, instead of teaching verbalizations would it not be better to teach generalizations and meaning? Many of us teach division as a step by step process: divide, multiply, subtract, and bring down. Instead why not use generalization such as: (1) When the dividend is constant, the value of the quotient varies inversely with the value of the divisor. (2) The quotient decreases correspondingly with decreases in the divisor. (3) The divisor multiplied by the quotient equals the dividend. (4) The divisor and quotient are related as multiplier and multiplicant are related to the product in multiplication. (5) When the divisor is constant the quotient is directly proportional to the value of the dividend. (6) When the value of the quotient is constant, the divisor and dividend may vary in direct proportion.

Could not these generalizations replace the step by step method of teaching short or long division?

Could not we then say that we are upholding one of the objectives of education to help children develop an understanding and appreciation of our number system? Yes, I was wondering if this could not be considered by educators as a small bit of food for thought.

Contributed by
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Measuring the Meanings of Arithmetic

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The meaningful method of teaching arithmetic is generally accepted by most teachers and professional educators, but to the best of the writer's knowledge few, if any, attempts have been made to measure the meanings of arithmetic. Standardized and teacher-made tests in the field of arithmetic usually measure abstract computational skills only. If we are to teach by the meaningful method then we should test to discover if the child comprehends the meanings and understandings inherent in our number system. The fact that a child can give the correct answer to the computation 25+38 indicates that he knows how to obtain the correct answer but does not necessarily indicate that he understands what he did to obtain the correct answer. However, if the child can obtain the correct answer to the following examples, it seems reasonable to conclude that he knows not only how to add but also understands what he is doing.

$$\begin{array}{r}
2 \text{ tens and } \ 5 \text{ ones} \\
+3 \text{ tens and } \ 8 \text{ ones} \\
\hline
__ \text{ tens and } __ \text{ ones, or } __
\end{array}$$

$$\begin{array}{rl}
572 & \text{when you carried the 1,} \\
+343 & \text{was it 1 one, 1 ten, or} \\
\hline
 & \text{1 hundred? } ____
\end{array}$$

Other examples to discover if a child has grasped the basic meanings and understandings of the addition process are as follows:

(1) Circle the dots below to show that 8+8+8+8 equals 3 groups of tens and 2 ones.

. . . .
. . . .

(2) 16 tens plus 3 tens and 4 ones plus 3 hundreds = ______

(3) Is 33+12 the same as 30+10+5? ______

(4) 29+7 are as many as 30 and ______

(5) Does 37+18 equal 40+15? ______

(6) In adding 698, 574, and 898, how many hundreds are carried? ______

(7) 13 tens and 17 ones equal 14 tens and ______ ones

(8) 14 tens plus 16 ones equal ______ tens and ______ ones

(9) Write the number that is 10 greater than 10,000 ______

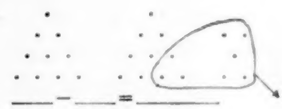
(10) 324 plus 6 tens = ______

Many children obtain the answers to subtraction problems by following a memorized routine procedure. For example, a child might be able to subtract 65 from 86 but not be able to subtract 5 tens and 15 ones from 8 tens and 6 ones. Following are some examples designed to discover if a child comprehends the meanings and understandings basic to the subtraction process:

---

(1) What subtractions do the following dot pictures show?

a. XXX
8 – ___ = _____

b.

___ – ___ = _____

(2) 6 tens and 5 ones
– 2 tens and 9 ones
__ tens and __ ones, or ___

(3) 439 When you borrowed the 1, was it 1 one, 1 ten, or 1 hundred? _____
– 256

(4) Would 120 – 35 be the same as 100 – 35 + 20? _____

(5) 7 tens and 8 ones are the same as 6 tens and _____ ones

(6) 8 tens and 6 ones are how many more than 5 tens and 7 ones? _____

(7) Write the number that is 5 less than 5,000 _____

(8) If 126 – 87 = 39, then 39 + _____ = 126

(9) If you change the 6 in 3600 to zero, how much smaller have you made the number? _____

The writer recently observed a fifth grade child multiply 36 by 47 using the following method:

```
   36
  ×47
   42   ( 7× 6=42)
  210   ( 7×30=210)
  240   (40× 6=240)
 1200   (40×30=1200)
 1692
```

We would all admit that the child used a round about method to obtain the correct answer, but the technique used by the child certainly shows that he has an understanding of the multiplication process. Following are some examples designed to discover if a child comprehends the meanings and understandings basic to the multiplication process:

---

(1) What multiplications do the following dot pictures show?

a.

6× ___ = _____

b.


(2) 3 tens and 5 ones
$\underline{\qquad\qquad\qquad\times 7}$
___ tens and ___ ones, or ___

(3) 160 When you carried the 4, was it 4 ones, 4 tens, or 4 hundreds?
$\underline{\times 8}$

(4) Does 19×5 equal $20\times 5-5$? ______

(5) $8\times 7=56$, or ______ tens and ______ ones

(6) Circle the dots below to show that $9\times 4=3$ tens and 6 ones

(7) What multiplications do the following additions show?

$45+45+45+45+45+45+45$ is $45\times$______ = ______

$90+60+30+60+90+120$ is $30\times$______ = ______

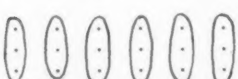
(8) Does 268×400 equal the sum of 200×400 plus 68×400? ______

(9)
```
   27
  ×32
  ---
  (9)4   Does the 9 mean 9 ones, 9 tens, or 9 hundreds? ______
 (8)1    Does the 8 mean 8 tens, 8 hundreds, or 8 thousands? ______
 ----
  904
```

The more difficult a process, the more necessary it is to test for meanings and understandings. Division is generally recognized as the most difficult of the four fundamental operations. This difficulty is partly due to the fact that before one can divide, he must be able to add, subtract, and multiply. If we have failed to teach and test for meanings adequately up to this point, the need for adequate evaluation becomes even more necessary. Following are some examples designed to discover if a child has developed the necessary meanings and understandings required to master the process of division.

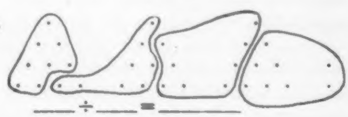
(1) What divisions do the following dot pictures show?

a.



18÷___=______

b.



___÷___=______

(2) ______ tens and ______ ones, or ______

$4 \,\overline{)\,12 \text{ tens and } 8 \text{ ones}}$

(3)
$$\begin{array}{r} 738 \\ 8\,\overline{)\,5904} \end{array}$$
Is the 7, 7 ones, 7 tens, or 7 hundreds? ______

(4) 12 tens÷3 ones=______ tens, or ______ ones
(5) Does 3600÷200 equal the sum of 3000÷200 plus 1600÷200? ______
(6) Write the subtraction below as a division

40−8=32−8=24−8=16−8=8−8=0 or 40÷______=______

(7)
```
       224
 36 | 8064
      (7)2      Does the 7 mean 7 hundreds, 7 thousands, or 7 ten thousands? ______
      (8)6      Does the 8 mean 8 tens, 8 hundreds, or 8 thousandths? ______
       72
      1(4)4     Does the 4 mean 4 tens, 4 hundreds, or 4 thousandths? ______
      144
```

(8) How many times can you take 300 from 15,000? ______

(9) 3600÷9=______ hundreds, or ______ tens, or ______ ones

If we are to teach arithmetic by the meaningful method then it is also necessary to test for meanings and understandings in arithmetic rather than for computational ability only. Testing for meanings and understandings will help to evaluate our teaching procedures in a more valid manner, since a valid test is one that is designed to measure the intended outcomes of instruction; and who would deny the acquisition of meanings and understandings as a cardinal objective or outcome of a good arithmetic program?

EDITOR'S NOTE. Dr. Koenker has illustrated the measurement of certain understandings of the meaningful structure of numbers and the way these are used in our conventional methods of calculation. He points out that the conventional procedures may be performed without a comprehension of what we are really doing with the numbers involved. His method is one of analysis of the structure of the numbers involved in the work so that it is possible to measure a given aspect of the whole procedure. Many teachers try to do this informally in their teaching so that their pupils have some comprehension of what is taking place as the numbers are manipulated. Failure to do this puts the child in the category of a computing machine. Both have their place in our society but for straight computation the machine usually excels the child. It is difficult to measure all of the types of understanding that are involved in a good program of elementary school mathematics. But we must not be in default. Rather let us take the challenge as did Dr. Koenker and see how far we can progress in the measurement of meaning and understanding.

Eat Your Numbers!

Many a time have we heard someone say, "I'll make you eat your words." For probably the first time a teacher can say, "Eat your numbers!" and the pupils like it. For arithmetic parties nothing is more "2th sum" than number cookies. Thanks to my brother who designed number cutters my pupils can now eat "wholesum" products.

Contributed by
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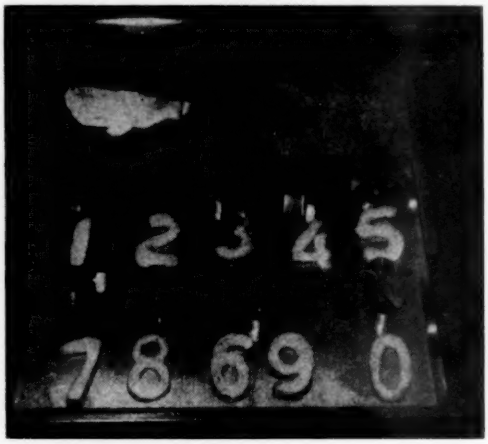

The Teaching of Roman Numerals

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The teaching of Roman numerals seems always to be minimized and occasionally skipped altogether in both elementary and secondary school mathematics. Is this current lack of attention to Roman numerals due to a decline in their importance—or is it possible that this lack of attention is due to the relative difficulty of teaching Roman numerals in relation to more conventional subject matter in mathematics?

The importance of teaching Roman numerals, it would seem, has neither increased nor decreased during recent years while other topics in mathematics have noticeably shifted in relative importance. It must be admitted that Roman numerals have maintained and will maintain a consistent position of importance in the historical background of mathematics—and that it is inconceivable that such a manifestly simple topic should be presented as a historical fact without an explanation of some aspects of the process involved. Further it may be observed that there has been no substantial decrease in the over-all number of popular uses of Roman numerals in recent years. Roman numberals provide a unique number system contrasting the common decimal system and serving to emphasize the vital importance of the concept of place value and the zero symbol in a way which cannot be duplicated by any other mathematics topic. Although numerous new ideas and topics have been added to the mathematics curriculum and attention devoted to and time spent upon traditional topics has decreased, there is no reason to discount the importance (admittedly not a tremendous importance) of the topic of Roman numerals.

Probably the one idea which causes by far the most confusion in the presentation of Roman numerals is the "subtraction" rule. Although it presents no difficulty as long as the pupil is confined to translating from proper Roman numerals into decimal numbers, a serious difficulty inevitably arises with a majority of pupils when they are asked in light of this rule to translate from decimal numbers into Roman numerals. For example, when pupils are asked to write 45 in Roman numerals, too often they will write down VL instead of XLV since the former is clearly a simpler statement than the latter. If the teacher holds to the "rule" approach, he will then be obliged to offer another rule, perhaps to the effect that certain of the numerals may never be "subtracted," namely, V, L, and D. This takes care of the writing of such numbers as 45, 95, 450, etc., but then the pupil may be asked to write 49 or 99 and an even larger number of pupils will write IL or IC rather then the correct XLIX or XCIX. The teacher holding to the "rule" approach then must either extend the new rule above (which could lead to other complications and additional confusion since obviously numerals such as I and X can be "subtracted") or he may devise another rule to cover these circumstances.

The teacher who has a valid understanding of Roman numerals himself soon learns to sidestep the "rule" approach of the textbooks and improvises his own technique of presentation. On the other hand the teacher with only a superficial understanding of this topic seemingly tends in one way or another to avoid its consideration. This avoidance may be manifested in a number of ways: the subject is simply skipped because it is considered relatively unimportant; only the small values are taught because these are the only numerals in frequent use; or the topic is presented following the available text with a minimal amount of time allowed

so that the class can proceed to another topic before complications arise. Unfortunately it also seems to be the case that some teachers having only a superficial understanding of Roman numerals do not recognize their shortcoming in this respect and proceed to teach authoritatively, leaving their pupils with misconceptions regarding this topic.

Much of the responsibility for this decline in attention to Roman numerals necessarily must be attributed to the producers of textbooks who apparently have effectively circumvented these difficulties rather than make any real effort toward their resolution. For example, a popular eighth grade textbook which devotes one page to Roman numerals explains them thusly:

> The rules for writing Roman numbers are as follows:
>
> A numeral following another numeral of equal or greater value adds its value . . .
>
> A numeral preceding one of greater value subtracts its value . . .
>
> A numeral between two numerals of greater value subtracts its value from the numeral following it, and the difference is added to the numeral preceding it . . .
>
> The numeral of larger denomination usually precedes the numeral of smaller denomination . . .
>
> When the form of addition is long, the shorter form of subtraction is preferred . . . .[1]

Obviously these "rules" are insufficient to define the Roman number system. Although other arithmetic textbooks offer a somewhat less confusing approach to this topic, too many of these textbooks leave much to be desired in their treatment of Roman numerals.

Although it is proper and common practice to present Roman numerals as they are used today, it is well to keep in mind that today's Roman numerals are slightly different from those used in the past. Consideration of the developmental changes in Roman numerals as well as consideration of the numerals of other ancient cultures would constitute an interesting enrichment topic, but this is clearly beyond the scope of the present paper.

It would seem to be among the easiest of tasks to teach Roman numerals in the following way, capitalizing upon the pupil's minimal understanding of the concept of place value derived from his acquaintance with the familiar decimal number system, and at the same time causing the pupil to appreciate more the importance and the value of the decimal system. Consider the table on the next page.

It may readily be observed from this table a) that the Roman numberals are formed by the arrangement of seven symbols (letters of the Roman alphabet), b) that the Romans, as demonstrated by their number system, had a tendency to think in terms of tens and multiples of ten, c) that they had no symbol to represent zero (and indeed there was no need for such a symbol in their system), and d) although what is seen here is the set of Roman numerals arranged in terms of the place values of the decimal system, it is clear that the Roman system existed without any regard for the idea of place value.

This table includes all proper groupings of the seven symbols of the Roman system (and the "bar" which represents one thousand times the value of the symbol with which it is used, i.e., $\overline{\text{V}}=5{,}000$ and $\overline{\text{C}}=100{,}000$). These groupings can be combined to form all of the numbers in the Roman system between one and 999,999, and of course, $\overline{\text{M}}$ would equal one million if one cared to go higher.

As an example of the use of this table, consider the problem of translating the decimal number 470,986 into Roman numerals. It is first necessary to recognize the decimal place values in the number, then it is a simple procedure to translate using the table as follows: take the four (fourth row) from the hundred thousands' column, then

[1] E. T. McSwain, Louis E. Ulrich, and Ralph J. Cooke, *Understanding Arithmetic 8* (Sacramento: California State Department of Education, 1957) p. 11. Omitted words indicated by ellipses were merely examples illustrating each of the five "rules."

TABLE I

ROMAN NUMERAL GROUPINGS ARRANGED WITH RESPECT TO THE PLACE VALUES OF THE DECIMAL NUMBER SYSTEM

	hundred-thousands'	ten-thousands'	thousands'	hundreds'	tens'	ones'	
(1)	$\overline{\mathrm{C}}$	$\overline{\mathrm{X}}$	M	C	X	I	(1)
(2)	$\overline{\mathrm{CC}}$	$\overline{\mathrm{XX}}$	MM	CC	XX	II	(2)
(3)	$\overline{\mathrm{CCC}}$	$\overline{\mathrm{XXX}}$	MMM	CCC	XXX	III	(3)
(4)	$\overline{\mathrm{CD}}$	$\overline{\mathrm{XL}}$	$\mathrm{M}\overline{\mathrm{V}}$	CD	XL	IV	(4)
(5)	$\overline{\mathrm{D}}$	$\overline{\mathrm{L}}$	$\overline{\mathrm{V}}$	D	L	V	(5)
(6)	$\overline{\mathrm{DC}}$	$\overline{\mathrm{LX}}$	$\overline{\mathrm{V}}\mathrm{M}$	DC	LX	VI	(6)
(7)	$\overline{\mathrm{DCC}}$	$\overline{\mathrm{LXX}}$	$\overline{\mathrm{V}}\mathrm{MM}$	DCC	LXX	VII	(7)
(8)	$\overline{\mathrm{DCCC}}$	$\overline{\mathrm{LXXX}}$	$\overline{\mathrm{V}}\mathrm{MMM}$	DCCC	LXXX	VIII	(8)
(9)	$\overline{\mathrm{CM}}$	$\overline{\mathrm{XC}}$	$\mathrm{M}\overline{\mathrm{X}}$	CM	XC	IX	(9)

add the seven (seventh row) from the ten thousands' column (since there is a zero in the thousands' place, nothing is added from the thousands' column) then add the nine from the hundreds' column, the eight from the tens' column, and the six from the ones' column. The equivalent Roman numeral thus would be $\overline{\mathrm{CDLXX}}\mathrm{CMLXXXVI}$ ($\overline{\mathrm{CD}}$*$\overline{\mathrm{LXX}}$**CM*LXXX*VI).

Obviously when teaching Roman numerals to less mature pupils the teacher may find it necessary to limit the presentation of the table to perhaps the ones', tens', and hundreds' columns. This would be particularly desirable where the teacher is introducing the topic for the first time and wants to avoid the possibility of such confusion as might result from the early introduction of the "bar" idea.

The principal merit of using a table of this kind as a technique for presenting Roman numerals is that it clearly demonstrates the consistency and regularity of the system and that in the table the pupil may readily observe the patterns of arrangement of symbols. The teacher may readily qualify the "subtraction" rule, pointing out that this rule would apply only to the fourth row and to the ninth row in each column. When the pupil can reproduce this table and when he understands that the symbols of the Roman system can be arranged in no other way than in the groupings which appear in this table, that only one grouping from each column may appear in a single numeral and that the grouping with the higher column (place) value always appears to the left of the grouping with the lower column value, he has mastered the topic.

EDITOR'S NOTE. The Roman numeral system like the British system of weights and measures grew somewhat like Topsy in that there was no close supervision or set of rules as development was taking place. Hence it is difficult to state a neat package of rules. Most scholars recognize changes in practice that came about over the centuries. It is interesting to note various forms of the Roman letters used. Particularly, the L, M, and D had interesting variations. And of course a high school freshman will recognize "duo-de-viginti" and its symbol IIXX. Our current standardization of Roman numerals is comparatively recent. Books of a little more than a century ago frequently separated the groupings of Roman numerals with a dot so that each group would represent one Hindu-Arabic value.

What restriction should be placed on translating from one number system to another? For the present, would a few years beyond the current 1960 be acceptable? Is it worthwhile for the more able students to look into the principles of number systems other than our Hindu-Arabic? Is there anything wrong with studying the history of mathematics and science as well as the history of wars, ideologies, and the movements of peoples?

Teaching Square Root Meaningfully in Grade 8

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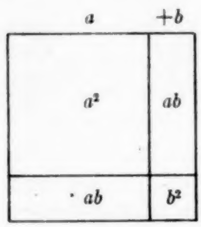
SQUARE ROOT CAN BEST BE TAUGHT at the eighth grade level as part of the unit on the area of rectangles (and squares). The system though basically the same as the expansion of $(a+b)^2$ taught in ninth grade, leads to quicker and more complete comprehension of the square root algorithm. The diagrams which follow will illustrate both the similarity and the differences.

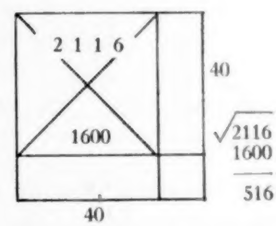
The square root part of the unit should be prefaced by careful work on $A=lw$ and $A=S^2$. That work should then be followed by $l=A/w$ or $w=A/l$ and that in turn by $S=\sqrt{A}$, where A comprises two-digit squares with which pupils are thoroughly familiar. The object here is to obtain mastery of such facts as $7^2=49$ and $\sqrt{18}=9$, and the idea that these are inverse operations in exactly the same way as multiplication and division.

The next step should deal with relationships like $30^2=900$, $90^2=8{,}100$, and $\sqrt{6{,}400}=80$. From $10^2=100$ and $99^2=9{,}801$ it can be pointed out that any three or four digit square will have a two digit root.

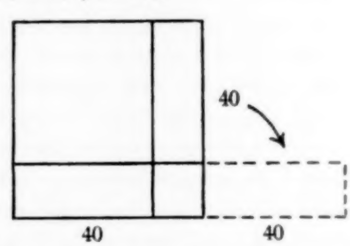
We are now ready to start actual extracting of square roots. Pupils will recognize that if we wish to find the square root of a number like 2,116 we may think of it and can draw it as a square. (Note the similarity with the usual system.) By means of a previously presented table of decade squares from 10^2 through 90^2 they see that the largest square which can be taken from 2,116 by observation is 1,600. They can also see that the remaining area is 516.

Since the square which was taken out is 1,600, the side of that square is 40. Inspection shows that the remaining area consists of two equal rectangles, each 40 long, and a small square, the side of which, although un-





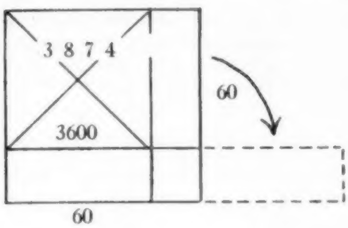
known, must be less than 10, and equal to the width of each rectangle. If we rearrange these three spaces as shown in the next diagram, so that they form one long rectangle, we know that the length of this new rectangle is something more than 80 but less than 90 and that the area equals 516. Since we have an approximate length and the area of the rectangle, we find the approximate width by $w=A/l=516/80=6$ (approx.). But having found the approximate width, observation shows that this width is the side of the small square and so must be added to the length of the large rectangle. We now have a more exact length, so we redivide (by 86 rather than 80), and get $w=516/86=6$. Multiplying 6×86 ($A=lw$) we get 516; thus $\sqrt{21'16}=40+6=46$.



When this process has been repeated many times, we begin to develop rules. With the use of our table of expanded squares we show the reason for marking off in pairs, beginning at the decimal point: $20^2=400$, $80^2=6{,}400$, $.12^2=.0144$, etc. In our original

example we used the 40 twice in our trial divisor because there were two rectangles; we used the 6 once because the side of the small square was used but once in the length of the large rectangle. We multiplied 86×6, because we were finding the area of a rectangle. Finally we used 40 for the length of each small rectangle rather than 4 because we were taking 1,600, not 16, from the original square, 2.116.

Now for numbers which are not perfect squares, for example, 3,874! Using the system already developed, we have:

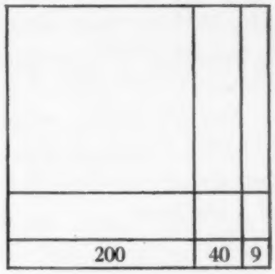


$$\begin{array}{r|r} & 60+2 \\ & \sqrt{38'74} \\ & 36\ 00 \\ \hline 120 & 2\ 74 \\ 122 & 2\ 44 \\ \hline & 30 \end{array}$$

or in our last step a remainder of 30 with a divisor of 122. Temporarily we may let pupils assume this as a fraction 30/122, which is obviously less than ½, although it would be well to point out later that a more exact fraction is 30/124 or 30/2×62. Therefore our root to the nearest whole number is 62. This may be followed by trying such a number as 5,884, where we have a reminder of 108 with a divisor of 146, or a fraction clearly more than ½. The root to the nearest whole number is 77, and again the more exact fraction, 108/152, can be emphasized as soon as pupils are ready.

$$\begin{array}{r|r} & 70+6 \\ & \sqrt{58'84} \\ & 49\ 00 \\ \hline 140 & 9\ 84 \\ 146 & 8\ 76 \\ \hline & 1\ 08 \end{array}$$

After several such problems to the nearest whole number (always selected so that the remainder, expressed as a fraction, is considerably more or less than ½) we are ready for squares of five or six, and later more, digits. If we try 62,001, we may again show



$$\begin{array}{r|r} & 200+40+9 \\ & \sqrt{6'2,0'01} \\ & 4\ 00\ 00 \\ \hline 400 & 2\ 20\ 01 \\ 440 & 1\ 76\ 00 \\ \hline 480 & 44\ 01 \\ 489 & 44\ 01 \end{array}$$

it graphically as a square from which we take successive increments.

Several such numbers will give enough back-ground to introduce the usual algorithm:

$$\begin{array}{c} 4\ \ 2\ \ 6 \\ \sqrt{18'14'76}, \end{array}$$

eliminating the additions in the root, and emphasizing that the zero is assumed to be there at any successive step, for the purpose of establishing the trial divisor, to represent the next group (pair) with which we are concerned. We can also note the pitfalls in such numbers as $\sqrt{64'06'4,0'16}, = 8{,}004$, where the zeros must be properly entered in both root and trial divisor, or $\sqrt{23'97}$. The last is particularly worth noting because of the dissimilarity with division, i.e., a remainder larger than the divisor is not necessarily incorrect. Here the increased divisor is even more apropos, 48 93/96 being less than 49.

$$\begin{array}{r|r} & 4\ \ 8 \\ & \sqrt{23'97} \\ & 16 \\ \hline 80 & 7\ 97 \\ 88 & 7\ 04 \\ \hline & 93 \end{array}$$

Finally we are ready for irrational roots as exact as we wish. Extending the algorithm to several decimal places for such roots as $\sqrt{2}$ or $\sqrt{3}$ is easily demonstrated. These should be followed by specially chosen numbers such as $713{,}180 < (844)^2 + 844)$.

	8 4 4
	$\sqrt{71'31'80}$
	64
160	7 31
164	6 56
1680	75 80
1684	67 36
	8 44

844/1684 is more than ½ and 844/1688 exactly equals it, but extension of the algorithm gives 844.499+. Clearly we may not always use the idea of a fraction.

We are now in position to show informally but clearly why $\sqrt{2}$ cannot ever "come out evenly." If it did we would necessarily have a digit in the trial divisor which, multiplied by itself, would give a product ending in zero. Of course, this does not preclude a repeating decimal and therefore an exact fraction, but the need for introducing the question will scarcely arise here with eighth graders.

EDITOR'S NOTE. Why teach square root? Isn't this in the same category as the "dead languages." Yes, it is in the same category but the usual "dead languages" are dead only to those who have little intellectual sophistication and discrimination. Square root is a part of mathematics in its relationship to squaring. It is also a useful concept in many formulas as for example, $s=\sqrt{A}$, $r=\sqrt{A/\pi}$ and the many more complex formulas in physics and engineering. Whether the square root process should be taught in grade eight to all pupils, to some, or to none is an open issue. How the process should be learned is also debatable. Some argue for teaching a method of approximation based upon division. Others want the Euclidean method learned immediately and some ask that the extraction of all roots be delayed until logarithms are available and the method can be more general. The editor learned the process of cube root as a youth and later has had some students who "discovered" cube root by analysis of $(a+b)^3=a^3+3a^2b+3ab^2+b^3$. Try Mr. DeGraff's method. How do you like it?

A Report on the Use of Calculators

Lois L. Beck
Riverside City Schools, California

Several fourth, fifth, and sixth grade classrooms in the Riverside City Schools are participating in an experimental program guided by Dr. Arden Rudell, Professor of Education, University of California at Riverside and Mr. Lewis Wickens, Director of Elementary Education. The administrators and teachers involved in this experiment are attempting to determine the extent to which use of the calculator may stimulate learning of arithmetic skills by children in the upper elementary grades. This experiment is being conducted as a part of the National Defense Education Program.

Monroe Educator Calculators are being used in the experiment. They are a light weight, manually operated computing machine about the size of a portable typewriter. The Monroe Educator mechanically performs addition, subtraction, multiplication and division in much the same manner as these basic operations are done with pencil and paper.

How does use of the calculator stimulate learning in the elementary arithmetic curriculum? This is the prime question with which we are now confronted in evaluating the utilization of the calculator in several experimental intermediate grades. We believe the calculator is valuable as a teaching tool in the intermediate grades; but, at the present time we are not certain how much it improves learning, when it should be introduced, or what methods and materials are most effective.

Although it is too early for results, here are some observations. The pilot classrooms are demonstrating that elementary school children can readily learn to operate the calculators and, when used as a regular classroom tool, tend to motivate and reinforce understanding and achievement in basic arithmetic skills. All the children I have observed using the calculators seem to enjoy this. They are enthusiastic about arithmetic. When you get a child interested in something, he will usually learn it. More important, the calculator demonstrates the basic principles of arithmetic. For example, the student learns from the machine that there are only two basic processes—addition and subtraction, and that multiplication and division are merely a series of one of these. The concept of "place value" is reinforced when students learn to place three and four digit numbers in the correct columns on the keyboard. These arithmetical rules are practiced in going through the mechanical operations of using the machine.

Besides stimulating children's interest and making arithmetic more meaningful, the use of the calculator seems to foster better work habits. It helps to develop habits of accuracy and neatness, wise use of time, checking work, attentiveness and concentration. It also helps students to follow oral and written directions, care for materials, and cooperate with others. The calculator, where I have observed its use, is regarded by the teachers as a valuable educational tool. The students look upon it with marked favor and enjoy the opportunity to use it.

Because of the apparent advantages of the calculator in reinforcing and extending arithmetic skills, the Riverside City School District is making more of these machines available for use in elementary classrooms.

Annual Business Meeting

Notice is hereby given, as required by the Bylaws, that the Annual Business Meeting of the National Council of Teachers of Mathematics will be held at the Statler Hilton Hotel, Buffalo, New York, at 4:00 p.m., April 22, 1960.

M. H. Ahrendt, *Executive Secretary*

The Abacus and our Ancestors

ROBERT W. FLEWELLING
Contra Costa County Schools, Martinez, Calif.

ALTHOUGH NOT GENERALLY APPRECIATED, the history of the abacus in the development of western civilization makes an interesting and effective vehicle for integrating mathematics and social studies in the classroom. The average American, if he knows about it at all, probably thinks of the abacus as a toy for children or an antiquated gadget that is used by some Orientals to "figure with." Many teachers are little better informed even though abacuses are common in elementary classrooms.

It comes as a surprise to many people to be told that, although the abacus is indeed primitive, it is nonetheless a very efficient tool in the hands of a skillful operator. Several times in recent years a race has been arranged between an expert with a mechanical computer and an expert with the abacus. In many of these contests, the abacus operator proved to be the faster!

Some are also surprised to learn that there are undoubtedly more business calculations performed on the abacus today than are performed on mechanical computers. Actually a very high percentage of the Asiatic population of the world still uses the abacus almost exclusively to solve everyday problems in arithmetic.

Our western heritage reflects a form of the abacus in that the board over which business was transacted in ancient markets had parallel grooves carved into its surface. Seeds, pebbles, or similar objects were manipulated in these grooves to find the sum of prices paid for various items of merchandise. This board was therefore called a counter. Our modern supermarket no longer uses grooves in a board for a "cash register," but we still retain a symbol of the system when we conduct our business over the "counter."

Contrary to popular conception, the abacus is not considered a Chinese invention. Some historians give credit to a Near Eastern or East Indian source.

As a matter of fact, the abacus, like most important basic inventions, is a comparatively simple contrivance. In essence, it is no more than a compact tallying device in which place value permits repetition of a few basic symbols to represent numbers of any practical magnitude.

If we think of the abacus as any such device, then it can be demonstrated that the abacus was independently invented in more than one primitive culture.

For example, it is virtually certain that the invention was made independently by the Incas in the New World. Their *quipus* were an interesting variation of the abacus in which the tallies were made with knots in cords—the location of the knots indicating place value significance.

It is also apparent that the use of tallies of one sort or another is the oldest root on the tree of number science. It is highly probable that tallying predates the earliest use of numbers as abstract symbols. Tallying is common today among peoples who make little use of abstract numbers.

Even in the primary classrooms of our own country, it is not uncommon to see a child using a one-to-one relationship in "counting" the number of sheets of paper or pencils for a group of children when there is insecurity in the use of abstract numbers for the purpose. This is essentially a form of tallying.

Although the full story of the development of the abacus can never be known, it is tempting to try to imagine how such an invention might have come about:

In his most primitive state, man was a

hunter and a food gatherer. Societies existing at this level have little or no need for numbers. There are, in fact, primitive groups of this type still in existence who use a number system consisting of one, two, and many.

The explanation is simple; such people must be able to move freely to follow their sources of food. They have no need to count because they can accumulate nothing in quantity. Their belongings must be few, simple, and portable.

Such a culture results in cycles of feast and famine, because livelihood is entirely dependent on the success of the hunt. We know that rounding up whole herds of animals and driving them over a precipice, into a morass, or into a natural or manmade enclosure for slaughter was one of the methods used by primitive groups to procure food. This technique was wasteful because it sometimes provided much more food than could be eaten, and methods of preservation were limited if they existed at all.

One day, we can imagine, a band of hunters may have trapped a herd alive. As they were about to proceed with wholesale slaughter, some deep thinker conceived the idea that keeping some animals alive in confinement would prolong the food supply. Meat on the hoof does not spoil.

This little group of our ancestors was then freed temporarily from the incessant pressure of the immediate need to search for food. Such a condition was pleasant and desirable, so at intervals the process was repeated.

Perhaps hundreds of years later another band of hunters, employing this "new" method of hunting, happened to use an enclosure where grass was plentiful and water was available. Quite by chance they may have rounded up enough animals that their need for food did not exceed the reproductive capacity of the herd. The band was thus freed permanently from the need to hunt. This must have seemed a superior way of life indeed to people who had been accustomed to living through periodic times of hunger!

In time the confined animals became tame enough to be herded. Then, when the grass or water supply became inadequate, part of the herd could be driven out to other pastures. Mankind had entered a new era. Some men became herdsmen instead ot hunters.

The impact of such a change must have been tremendous in its effect on the society and culture of mankind. Men now had a new freedom to think, to dream, and to create. More human beings could live together than was possible in a food gathering culture. As men learned to live in larger groups, specialization became possible, and a new division of work was called for.

With this new mode of life came a new need. One does not manage a herd effectively without some system of keeping track of the individual animals in the herd. We can imagine a primitive herdsman keeping a record of the cattle taken from the corral in the morning by use of a method that is still used by some peoples. He simply placed a pebble on a pile for each animal as it went out. When the herd was returned, a pebble was removed for each animal as it came in.

Note that it was not necessary for our herdsman to known *how many* animals *he had*. His concern was only that *all* his animals *were there*. He was using a one-to-one relationship and no real concept of numbers was necessary at all.

The next step historically is obvious. Man had very convenient built-in counters. It was much easier to "tally off" the fingers of one hand, or of both hands, or of both hands and both feet before placing a stone on the pile. Numbers probably were developed in naming the fingers and toes while counting them off. The names of the extremities are found in number systems in languages all over the world. Our own word *digit* meant finger in Latin.

Another essential in the development of number systems is often taken for granted. This was the discovery of the process of simplification by periodic repetition of the basic symbols. As the symbols were repeated, the repetitions were tallied in turn in another "place."

Probably counters were arranged in groups corresponding to the basic concepts of place value early in the development of number systems. For example, in the decimal system, groups of ten pebbles, sticks, or like objects were used, and, as each pile was exhausted, the count was carried as one in the pile of next higher place value. This system of counting was an abacus; it was also the root and heartwood of our number system.

If number sense developed in this way (and at least we can say that it could have), then the basic characteristics of the abacus were a necessary foundation even for the invention of numbers themselves.

We have indulged our fancy in trying to reconstruct what may have happened in the beginning of number science. We have no need for such speculation to find that the abacus has made vital contributions in recorded history. Until the Hindu-Arabic system of notation, with its indispensable zero, was invented, number language was used only to record the results of computation. Nearly all solutions involving arithmetic were found with the aid of some form of abacus.

It was the ability to reproduce a number as it is indicated on the abacus by place values assigned to recurring basic symbols that made the Hindu-Arabic system of notation such a revolutionary development in the history of mathematics. The invention of zero to indicate a place that was not occupied by counters on the abacus was vital to the system. To the modern mind, in retrospect, the necessity for an indicator such as the zero seems ridiculously obvious. That it evidently was not, is indicated by the fact that some historians have classified the zero and the wheel as the most important inventions of mankind.

Although our culture has outgrown the need to use the abacus for calculation, some understanding of its functioning contributes a great deal to our comprehension of our number system and the basic arithmetic processes. The abacus is especially useful in developing understandings of place value, the significance of zero, and decimal fractions.

In addition to this significance for understanding of number science, a little research into the history of the abacus can contribute a great deal to our understanding of the moulding of our culture which resulted from the application of number science in numeration, measurement, analysis, and prediction. The refinement of the tools of number science was at least an important part of the growth of civilization—it might even be said to have been the foundation. And the history of the abacus, more than any other development, presents a graphic picture of man's progress in working and thinking with numbers.

EDITOR'S NOTE. The author must necessarily surmise on events that led to the development and use of various types of counting frames. The store "counter" and the word "calculate" are associated with ancient forms of reckoning with the use of a type of abacus. Many teachers have used a counting frame and these have ranged from clothes pins on an old coat hanger to elaborate frames with colored markers. The value of these depends so much upon how they are used. They can be genuinely educational or they can be only "playing with beads." Let us strive for the former and limit their use to situations where the understanding of arithmetic is enhanced.

A New Editor for The Arithmetic Teacher

After June first, 1960 Professor E. Glenadine Gibb of Iowa State Teachers College, Cedar Falls, Iowa will be the editor of this journal. All materials and manuscripts should be sent to her after March First.

Dr. Gibb is very well known for her writing, speaking, and researches in the teaching of arithmetic. For the past five years she has served as an associate editor of THE ARITHMETIC TEACHER. With her direction the journal will continue to grow and serve the schools in the important years ahead.

National Council of Teachers of Mathematics

Spring Meeting—April 21–23, 1960
Hotel Statler Hilton, Buffalo, New York

The Thirty-Eighth Annual Meeting of the National Council of Teachers of Mathematics will be held in the Hotel Statler Hilton, Buffalo, New York on Thursday, Friday and Saturday, April 21–23, 1960. Registration will begin Wednesday afternoon, April 20.

The opening general session address "Mathematics and Human Knowledge" will be made by Dr. Carroll V. Newsom, President of New York University. At this Thursday evening session, Dr. H. Vernon Price of the State University of Iowa will preside. The Twenty-Fifth Yearbook of the National Council of Teachers of Mathematics will be presented.

The Friday evening general session will feature the Annual Banquet presided over by Dr. H. Glenn Ayre of Western Illinois University.

Elementary school teachers will be able to hear Dr. J. Fred Weaver, Dr. Robert L. Swain, Dr. Ann C. Peters, Dr. Charlotte W. Junge, Dr. John L. Marks, Dr. David A. Page, Dr. William B. Higgins and Dr. W. Warwick Sawyer and other leaders. Topics to be discussed will include: Algebra in Grades 5–8; Definitions in Arithmetic, Depth in Arithmetic Learning; Teaching the Language of Per Cent; Ideas for the Elementary School; Set, Scale and Symbol and the Elementary Mathematics Curriculum, etc.

At the Junior High School level, talks will be presented by Kenneth C. Skeen, Eugene P. Smith, Doris McLennan, Helen L. Garstens, Paul C. Rosenbloom, Robert B. Davis, R. D. Anderson, George Schaefer and others. Topics include: Occupation and Math Courses; Laboratory Mathematics; The Madison Project; Experimentation in Math Curriculum, Non-Metric Geometry.

The Senior High School sections will feature: A Seminar in Modern Mathematics; Learning Mathematics by Solving Problems; Vector Trigonometry and Updating Mathematics Instruction, etc. The Saturday morning session of this group will feature "What I Saw in Russian Classrooms," a presentation by Robert E. Rourke of the Kent School.

College teachers will be interested in hearing E. P. Vance on "The Advanced Placement Program in College"; Myron F. Rosskopf on "Teacher Training in Norway" and Paul Johnson on "Cooperative Institutes Interpret Modern Mathematics."

BOOK REVIEW

Dimensions, Units, and Numbers, in the teaching of physical science, Renee G. Ford and Ralph E. Cullman (New York: Bureau of Publications, Teachers College, Columbia University, 1959). Paper, ix+49 pp., $1.00.

This pamphlet, one of five prepared by the Science Manpower Project, was written for the science teachers of the secondary schools. There are some sections that may be of interest to mathematics teachers of the elementary and secondary schools.

The first part, Dimensions, pertains to the dimensional analysis used by the teachers in their work with physical formulas. Here, the mathematics teacher can find many examples of integral exponents, negative and zero as well as positive.

The second part, Units, contains a discussion of unit analysis, a method which permits the operation on physical units as if they were algebraic terms. This section in-

cludes a good exposition, with many examples, of the problems of conversion of units. The explanation may be helpful to teachers of arithmetic and would help their students decide on the process, multiplication or division, to be used in converting from one system of measures to another or from one unit to another in the same system.

The final part, Numbers, is the most pertinent for teachers of mathematics. This section includes discussions and examples of scientific notation, approximate numbers, ratio, and variation and proportion.

There are several statements which cannot be accepted by teachers of mathematics. The first of these, page 14, determines the dimensions of G in the equation $F=Gmm'/r^2$ by use of the principle of dimensional homogeneity. The authors then substitute the dimensions of G in the original dimensional equation and state that the dimensions of G satisfy the principle of dimensional homogeneity. This is merely a check of the steps in the computation. It does not constitute proof that the dimensions of G satisfy the principle of dimensional homogeneity.

A second error, page 25, which is difficult to overlook, but may be excused since the authors may not be sports-minded, concerns a runner who covers a distance of 200 yards in 15 seconds. This is shown to be equivalent to 2.73 miles per hour. The correct answer is obtained by multiplying this result by 10, which would be the answer obtained from the data and formula if correctly completed.

In the handling of scientific notation, the discussion of products and quotients is treated properly when compared to the latest mathematics texts. However, the notion of precision is confused with the notion of significance in the addition and subtraction of approximate data.

In rounding off approximate data, the authors use the older method of "rounding up" the digit preceding the discarded digit, if the discarded digit is "equal to or greater than five." The newer method, based on probability and used in the field of engineering, states that if the discarded digit is equal to five, one is added to the preceding digit if the preceding digit is odd but the preceding digit remains the same if it is even. No mention is made of this method in spite of its general usage.

The pamphlet is readable and may satisfy a felt need in the science field, but it adds little to the mathematics field. This reviewer would not recommend that the pamphlet be purchased unless the reader is looking for some applications from science to illustrate his work in mathematics.—*John C. Bryan, State University of New York, College of Education, Cortland, New York.*

THE PERFECT SQUELCH

Mr. Brown was an impressive-looking man who appeared able and willing to achieve almost anything. To hear him talk, he could build cities, move mountains and change history. His inability to control his only son, a spoiled-brat replica of himself, was notorious in his neighborhood, however. The boy did entirely as he pleased, and his school grades soon reflected this. Mr. Brown, unable to believe that his darling boy had failed in every subject, stormed into the school principal's office. Held at bay by a little, gray-haired secretary, he slammed his boy's report card on her desk.

"If I can't see the principal immediately," he snorted, "just who do I start talking to about my son's grades?"

The little secretary, unawed, said pleasantly: "To your son."

H. T. HENDREN